Lectures on the Theory of Reciprocants.

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[Reported by JAMES HAMMOND, M. A.]

LECTURE XXV.

In a letter to me dated June 14th, 1886, M. Halphen calls forms which are persistent under the substitution $\frac{1}{x}$, $\frac{y}{x}$, *Invariants d'homologie.* He uses the letters
$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n$$
to denote y and its successive modified derivatives with respect to x; and, supposing them to become
$$A_0,\ A_1,\ A_2,\ A_3,\ \ldots\ A_n$$
in consequence of the substitution $\frac{1}{x}$, $\frac{y}{x}$, gives, in the briefest possible manner, two very ingenious proofs of the formula
$$A_n = (-)^n x^{2n-1}\left\{a_n + \frac{n-2}{1.x}a_{n-1} + \frac{(n-2)(n-3)}{1.2.x^2}a_{n-2} + \ldots\right\},$$
from which he deduces the theorem that the substitution in question changes any homogeneous and isobaric function f, of degree i and weight ω in
$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_n,$$
into
$$F = (-)^\omega x^{2\omega-i}e^{\frac{\Theta}{x}}f,$$
where Θ is the partial differential operator
$$-a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + \ldots + (n-2)\,a_{n-1}\partial_{a_n}.$$

I give the two proofs mentioned above in M. Halphen's own words, adding occasional footnotes, and making slight changes in the literation of his formulae when it seems desirable to do so.

Soient
$$X = \frac{1}{x},\ Y = \frac{y}{x}.$$

Par une formule connue (Schlömilch, Compendium II.)

$$\frac{d^n y}{dX^n} = (-1)^n x^{n+1} \frac{d^n}{dx^n}(x^{n-1}y)^*$$

et puisque $Y = Xy$, il en resulte

$$\frac{d^n Y}{dX^n} = X\frac{d^n y}{dX^n} + n\frac{d^{n-1}y}{dX^{n-1}} = (-1)^n x^n \left\{\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y)\right\}$$

$$= (-1)^n x^{2n-1}\left\{y_n + \frac{n(n-2)}{1.x}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2.x^2}y_{n-2} + \ldots\right\}^\dagger$$

Si l'on pose

$$\frac{d^n Y}{dX^n} = n!\,A_n,\quad y_n = n!\,a_n$$

il vient

$$\text{(I)}\qquad A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{n-2}{1.x}a_{n-1} + \frac{(n-2)(n-3)}{1.2.x^2}a_{n-2} + \ldots\right\}.$$

Soit

$$\Theta f = \Sigma\,(n-2)\,a_{n-1}\frac{\partial f}{\partial a_n}{}^\ddagger$$

on aura

$$\Theta a_n = (n-2)\,a_{n-1},$$
$$\Theta^2 a_n = (n-2)(n-3)\,a_{n-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_n = (-1)^n x^{2n-1}\left\{a_n + \frac{1}{1.x}\Theta a_n + \frac{1}{1.2.x^2}\Theta^2 a_n + \ldots\right\}.$$

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* An easy inductive proof of this may be obtained as follows:

Since $\frac{d}{dX} = -x^2\frac{d}{dx}$ we have $\frac{d^{\kappa+1}y}{dX^{\kappa+1}} = -x^2\frac{d}{dx}\left(\frac{d^\kappa y}{dX^\kappa}\right)$.

Hence, assuming the truth of the formula when $n=\kappa$, we find

$$\frac{d^{\kappa+1}y}{dX^{\kappa+1}} = (-)^{\kappa+1}x^2\frac{d}{dx}\left\{x^{\kappa+1}\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^2\left\{x^{\kappa+1}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\,x^\kappa\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^{\kappa+2}\left\{x\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^{\kappa-1}y) + (\kappa+1)\frac{d^\kappa}{dx^\kappa}(x^{\kappa-1}y)\right\}$$
$$= (-)^{\kappa+1}x^{\kappa+2}\frac{d^{\kappa+1}}{dx^{\kappa+1}}(x^\kappa y).$$

Thus, if the formula is true for $n=\kappa$, it will be equally so when $n=\kappa+1$. But it is obviously true when $n=1$ $\left(\text{when it becomes } \frac{dy}{dX} = -x^2\frac{dy}{dx}\right)$, and therefore holds universally.

† For, expanding by Leibnitz's Theorem,

$$\frac{d^n}{dx^n}(x^{n-1}y) - n\frac{d^{n-1}}{dx^{n-1}}(x^{n-2}y) = x^{n-1}y_n + n(n-1)\,x^{n-2}y_{n-1} + \frac{n(n-1)}{1.2}(n-1)(n-2)\,x^{n-3}y_{n-2} + \ldots$$
$$-n\{x^{n-2}y_{n-1} + (n-1)(n-2)\,x^{n-3}y_{n-2} + \ldots\}$$
$$= x^{n-1}y_n + n(n-2)\,x^{n-2}y_{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2}x^{n-3}y_{n-2} + \ldots$$

‡ The summation extending to all positive integral values of n, from 1 to ∞, so that

$$\Theta = -a_0\partial_{a_1} + a_2\partial_{a_3} + 2a_3\partial_{a_4} + 3a_4\partial_{a_5} + \ldots$$

Remembering that Halphen's $a_0, a_1, a_2, a_3, \ldots$ have the same meaning as our $y, t, a, b, \ldots$, this operator is $-y\partial_t + a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$ identical with the Θ used in previous lectures.

Par conséquent, pour une fonction contenant $a_0, a_1, a_2, \ldots,$ de degré i et de poids ω, à chaque terme, on aura

$$F = (-1)^\omega x^{2\omega - i}\left\{f + \frac{1}{1.x}\Theta f + \frac{1}{1.2.x^2}\Theta^2 f + \ldots\right\}.^*$$

C. Q. F. D.

Autre Demonstration de la Formule (I).†

Si l'on change X et x en $X+H$ et $x+h$, on a

$$h = -\frac{H}{X(X+H)}.$$

Maintenant la formule

$$y = a_0 + ha_1 + h^2 a_2 + \ldots + h^n a_n + \ldots$$

écrite *symboliquement* ‡

$$y = \frac{1}{1 - ah}$$

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* We may show without much difficulty that, when $\Theta_1, \Theta_2, \Theta_3, \ldots$ are each of them equivalent to Θ, but Θ_1 acts on u only, Θ_2 on v, Θ_3 on w, and so on, $\Theta uvw \ldots = (\Theta_1 + \Theta_2 + \Theta_3 + \ldots) uvw \ldots$ From this it can be deduced that $\Theta^\kappa uvw \ldots = (\Theta_1 + \Theta_2 + \Theta_3 + \ldots)^\kappa uvw \ldots$, when κ is any positive integer. Now let the number of the functions $u, v, w, \ldots$ be i, and suppose that

$$u = a_n,\ v = a_p,\ w = a_q, \ldots;$$

suppose, also, that the weight $n + p + q + \ldots = \omega$. Then

$$A_n A_p A_q \ldots = (-)^\omega x^{2\omega - i}\left(e^{\frac{\Theta}{x}} a_n\right)\left(e^{\frac{\Theta}{x}} a_p\right)\left(e^{\frac{\Theta}{x}} a_q\right) \ldots = (-)^\omega x^{2\omega - i} e^{\frac{1}{x}(\Theta_1 + \Theta_2 + \Theta_3 + \ldots)} a_n a_p a_q \ldots$$

$$= (-)^\omega x^{2\omega - i} e^{\frac{\Theta}{x}} a_n a_p a_q \ldots$$

(for by what precedes $\Theta_1 + \Theta_2 + \Theta_3 + \ldots$ may be replaced by Θ). Taking $a_n a_p a_q \ldots$ and $A_n A_p A_q \ldots$ to be corresponding terms of f and F, we see at once that

$$F = (-)^\omega x^{2\omega - i} e^{\frac{\Theta}{x}} f.$$

† If x becomes $x + h$ in consequence of the augmentation of X by an arbitrary quantity H, the increment of x will not be a constant, but will depend on X as well as on H. The value of h may be found at once by eliminating x between $X = \frac{1}{x}$ and $X + H = \frac{1}{x+h}$, when we obtain $X + H = \frac{X}{1 + hX}$, and consequently $h = -\frac{H}{X(X+H)}$.

This increase of X also changes y and Y (functions of x and X, whose original values were a_0 and A_0 before the augmentation of X took place) into

$$y = a_0 + ha_1 + h^2 a_2 + \ldots + h^n a_n + \ldots$$

and into

$$Y = A_0 + HA_1 + H^2 A_2 + \ldots + H^n A_n + \ldots$$

These altered values of y and Y are the ones used in this second proof; the other letters retain their original signification.

‡ The word *symboliquement* indicates, whenever it is used, that powers of a are to be replaced by suffixes of corresponding value. *E. g.* in the final result $A_n = (-)^n x^{2n-1}\left(a^n + \frac{n-2}{x} a^{n-1} + \ldots\right)$ is to be replaced by $A_n = (-)^n x^{2n-1}\left(a_n + \frac{n-2}{x} a_{n-1} + \ldots\right)$.

In our notation the final result is $A_{n+2} = (-)^n x^{2n+3}\left(a, b, c, d, \ldots \between \frac{1}{x}, 1\right)^n$.

devient
$$y = \frac{X(X+H)}{X^2 + H(X+a)}.$$
D'ailleurs
$$Y = (X+H)\,y;$$
donc *symboliquement*

(II)
$$Y = \frac{X(X+H)^2}{X^2 + H(X+a)}.$$

Si l'on developpe le second membre (II) suivant les puissances ascendants de H, le coefficient de H^n est A_n. Or ce developpement est

$$Y = X\left\{1 + \left(1 - \frac{a}{X}\right)\frac{H}{X} + \left(\frac{a}{X}\right)^2\left(\frac{H}{X}\right)^2 + \ldots + (-1)^n\left(\frac{H}{X}\right)^n\left(1 + \frac{a}{X}\right)^{n-2}\left(\frac{a}{X}\right)^2 + \ldots\right\}$$

donc *symboliquement*

$$A_n = (-1)^n\,\frac{1}{X^{n+1}}\left(1 + \frac{a}{X}\right)^{n-2} a^2 = (-1)^n x^{2n-1}\left(a + \frac{1}{x}\right)^{n-2} a^2$$

ce qui est justement la formule (I).

We may regard the coefficients $a, b, c, \ldots$ of the ordinary binary Quantic in u, v,
$$(a, b, c, \ldots \between u, v)^n,$$
as the successive modified derivatives, beginning with the second, of a new variable y with respect to another new variable x.

Any invariant I of this Quantic will then retain its form unaltered, or at most merely acquire an extraneous factor, if

1° leaving x, y, v unaltered we change u into $u + \lambda v$,

2° " u, v " " " x, y " $\dfrac{x}{1+hx}, \dfrac{y}{1+hx}$,

3° " u, v " " " x, y " $\dfrac{1}{x}, \dfrac{y}{x}$,

where λ and h are arbitrary constants.

For we have seen that these three substitutions will severally convert any homogeneous and isobaric function F, of degree i and weight w in the letters $a, b, c, \ldots$, into
$$e^{\lambda\Omega}F,\ (1+hx)^{\nu}e^{\frac{h\Omega}{1+hx}}F, \text{ and } (-)^w x^{\nu} e^{\frac{\Omega}{x}}F,$$
where, in each case, $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$, and $\nu = 3i + 2w$. From our point of view an invariant is defined as a homogeneous and isobaric solution of the equation
$$\Omega I = 0.$$
Hence the above substitutions convert the invariant I into
$$I,\ (1+hx)^{\nu}I, \text{ and } (-)^w x^{\nu} I, \text{ respectively.}$$

An *absolute invariant* with respect to any substitution is one which, disregarding its sign, remains unchanged in absolute value by that substitution. Thus, any invariant for which

$$\nu = 3i + 2w = 0$$

is an absolute invariant with respect to each of the three substitutions here considered.

An invariant is of odd or even character with respect to any substitution according as its sign is or is not changed by that substitution. Thus, invariants are of odd or even character with respect to the substitution $\frac{1}{x}, \frac{y}{x}$ according as their *weights* are odd or even.

This corresponds to the theorem that the character (with respect to the interchange of x and y) of a pure reciprocant is odd or even according as its degree is odd or even (vide *American Journal of Mathematics*, Vol. VIII, p. 251).

From any two invariants for which ν has the same value we can form an absolute invariant (*i. e.* one for which $\nu = 0$) by taking their ratio, and then by differentiating the absolute invariant thus formed obtain another invariant.

Suppose I_1 to be an invariant of degree i_1 and weight w_1,
I_2 " " " " " " i_2 " " w_2,

and let $$3i_1 + 2w_1 = \nu_1, \; 3i_2 + 2w_2 = \nu_2;$$

then the ν for $I_1^{\nu_2}$ is the same as that for $I_2^{\nu_1}$, and consequently $I_1^{\nu_2} I_2^{-\nu_1}$ is an absolute invariant.

We proceed to show that $\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ is an invariant, though not an absolute one.

Using accents to denote differential derivation with respect to x, we have

$$\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1}) = I_1^{\nu_2 - 1} I_2^{-\nu_1 - 1}(\nu_2 I_1' I_2 - \nu_1 I_1 I_2').$$

If, then, we can prove that $\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$ is an invariant, it will follow that $\frac{d}{dx}(I_1^{\nu_2} I_2^{-\nu_1})$ will be one also, and the proposition will be established. It may be very easily shown that this is the case by using Cayley's generators P and Q. For (see *American Journal of Mathematics*, Vol. VIII, p. 221), I being any invariant of degree i and weight w, PI and QI are also invariants where

$$P = a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots) - ib,$$

and $$Q = a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots) - 2wb.$$

Hence $$(3P+Q)\,I \text{ is an invariant.}$$

Now, since $$3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots\ldots = \frac{d}{dx},$$

and $$3i + 2w = \nu,$$

$$(3P+Q)\,I = a\,(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots\ldots)\,I - (3i+2w)\,bI = aI' - \nu bI.$$

Consequently $$aI_1' - \nu_1 bI_1 \text{ and } aI_2' - \nu_2 bI_2$$

are both of them invariants. Hence the combination

$$\nu_2 I_2\,(aI_1' - \nu_1 bI_1) - \nu_1 I_1\,(aI_2' - \nu_2 bI_2) = a\,(\nu_2 I_1' I_2 - \nu_1 I_1 I_2')$$

is also an invariant; *i. e.*

$$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$$

is one; which is the theorem to be demonstrated.

The invariant $aI' - \nu bI$, which we generated from I, is of degree $i+1$ and weight $w+1$; its ν is therefore the original ν increased by 5 units, three for the unit increase in the degree and two for the unit increase in the weight. Hence, on repeating the process of generation, we obtain the invariant

$$\left\{a\frac{d}{dx} - (\nu+5)\,b\right\}(aI' - \nu bI) = a^2 I'' - 2\,(\nu+1)\,abI' - 4\nu acI + \nu\,(\nu+5)\,b^2 I.$$

By adding on the invariant $\nu\,(\nu+5)(ac-b^2)\,I$ and dividing the sum by a, the above invariant is reduced to

$$aI'' - 2\,(\nu+1)\,bI' + \nu\,(\nu+1)\,cI,$$

which is an invariant of lower degree by unity than the unreduced form.

The results obtained above may be compared with the corresponding ones in the theory of reciprocants.

Thus to the invariants	correspond the reciprocants
I (deg. i, wt. w),	R (deg. i, wt. w),
$aI' - \nu bI$,	$aR' - \mu bR$,
$\nu_2 I_1' I_2 - \nu_1 I_1 I_2'$,	$\mu_2 R_1' R_2 - \mu_1 R_1 R_2'$,
$aI'' - 2\,(\nu+1)\,bI' + \nu\,(\nu+1)\,cI$,	$5aR'' - 5\,(2\mu+1)\,bR' + 4\mu\,(\mu-1)\,cR$,
where $\nu = 3i + 2w$,	where $\mu = 3i + w$.

Defining a *plenarily absolute* form to be one whose degree and weight are both zero ($i=0$, $w=0$), the theorem I shall now prove may be stated as follows:

By differentiating a plenarily absolute principiant we obtain another principiant.

Let P be any principiant of degree i and weight w. Then, by what precedes, since P is both an invariant and a reciprocant,

$$a\frac{dP}{dx} - \nu bP \text{ is an invariant,}$$

and

$$a\frac{dP}{dx} - \mu bP \text{ is a reciprocant.}$$

Hence, when $\nu = 0$ (*i. e.* when $3i + 2w = 0$),

$$\frac{dP}{dx} \text{ is an invariant,}$$

and when $\mu = 0$ (*i. e.* when $3i + w = 0$),

$$\frac{dP}{dx} \text{ is a reciprocant.}$$

When both $\mu = 0$ and $\nu = 0$ (which happens when $i = 0$, $w = 0$),

$$\frac{dP}{dx} \text{ is both a reciprocant and an invariant;}$$

i. e.

$$\frac{dP}{dx} \text{ is a principiant.}$$

LECTURE XXVI.

In the theory of Invariants the annihilator Ω has two independent reversors any linear combination of which will also be a reversor. To each of these reversors there corresponds a generator for invariants. Thus Cayley's two generators

$$a(b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots\ldots) - ib,$$
$$a(c\partial_b + 2d\partial_c + 3e\partial_d + \ldots\ldots) - 2wb,$$

correspond to the two reversors

$$b\partial_a + c\partial_b + d\partial_c + e\partial_d + \ldots\ldots,$$
$$c\partial_b + 2d\partial_c + 3e\partial_d + \ldots\ldots$$

The only linear combination of these which does not increase the extent j as well as the weight of the operand is

$$O = jb\partial_a + (j-1)c\partial_b + (j-2)d\partial_c + \ldots\ldots$$

It is convenient to take this for one of our reversors, and for the other

$$\frac{d}{dx} = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots\ldots,$$

which is a reversor to V, the annihilator for reciprocants, as well as to Ω, the annihilator for invariants.

We saw in Lecture XI (*American Journal of Mathematics*, Vol. IX, p. 1) that when F is any homogeneous and isobaric function of degree i and weight w in the $j+1$ letters $a, b, c, \ldots.$

$$(\Omega O - O\Omega)F = (ij - 2w)F.$$

The method employed in proving this can also be applied to show that

$$\left(\Omega \frac{d}{dx} - \frac{d}{dx}\Omega\right)F = \nu F,$$

where $\nu = 3i + 2w$.

Corresponding to the reversors O and $\frac{d}{dx}$ we have the two generators for invariants

$$a\frac{d}{dx} - \nu b \text{ and } aO - (ij - 2w)b,$$

which are linear combinations of Cayley's generators.

Thus, if I be any invariant,

$$\left(a\frac{d}{dx} - \nu b\right)I \text{ and } \{aO - (ij - 2w)b\}I$$

are also invariants.

The operator $\frac{d}{dx}$ has, but O has not, analogous properties in the theory of Reciprocants; viz. $\frac{d}{dx}$ is a reversor to V and $a\frac{d}{dx} - \mu b$ is a generator for reciprocants. Thus, we have shown in previous lectures that

$$\left(V\frac{d}{dx} - \frac{d}{dx}V\right)F = 2\mu aF,$$

where F is any homogeneous and isobaric function, and $\mu = 3i + w$, and that if R is any pure reciprocant $\left(a\frac{d}{dx} - \mu b\right)R$ is one also.

Now, Mr. Hammond has found that if

$$W = \frac{b}{a}\partial_a + \frac{2ac - b^2}{a^2}\partial_b + \frac{3a^2d - 3abc + b^3}{a^3}\partial_c + \ldots.,$$

W is a reversor to V, and $a^2W - ib$ is a generator for pure reciprocants. In fact we have

$$\begin{aligned} VW - WV = {} & V\left(\frac{b}{a}\right)\partial_a \\ & + \left\{V\left(\frac{2ac - b^2}{a^2}\right) - W(2a^2)\right\}\partial_b \\ & + \left\{V\left(\frac{3a^2d - 3abc + b^3}{a^3}\right) - W(5ab)\right\}\partial_c \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

But, since

$$V\left(\frac{b}{a}\right) = 2a,$$

$$V\left(\frac{2ac-b^2}{a^2}\right) = 10b - 4b = 6b,$$

$$V\left(\frac{3a^2d-3abc+b^3}{a^3}\right) = \left(18c + 9\frac{b^2}{a}\right) - \left(15\frac{b^2}{a} + 6c\right) + 6\frac{b^2}{a} = 12c,$$

. . . .

and

$$W(2a^2) = 4b,$$

$$W(5ab) = 5\frac{b^2}{a} + 5\left(\frac{2ac-b^2}{a}\right) = 10c,$$

. . . .

it follows that

$$VW - WV = 2a\partial_a + 2b\partial_b + 2c\partial_c + \ldots\ldots = 2i.$$

Thus W is a reversor to V. Moreover, $a^2W - ib$ acting on any pure reciprocant generates another.

Let R be a pure reciprocant of degree i; then, by what precedes,

$$(VW - WV)R = 2iR.$$

But, since R is a pure reciprocant, $VR = 0$, and consequently $VWR = 2iR$. Now, $\quad V(a^2W - ib)R = a^2VWR - iRVb = a^2.2iR - iR.2a^2 = 0.$

Hence

$$(a^2W - ib)R$$

is a pure reciprocant; *i. e.* $\quad a^2W - ib$

is a generator for pure reciprocants.

Mr. Hammond shows that W is a reversor to V in the following manner:

Let

$$u = a_0 + a_1e^\theta + a_2e^{2\theta} + a_3e^{3\theta} + \ldots\ldots,$$

$$\phi(u) = A_0 + A_1e^\theta + A_2e^{2\theta} + A_3e^{3\theta} + \ldots\ldots,$$

$$\psi(u) = A_0' + A_1'e^\theta + A_2'e^{2\theta} + A_3'e^{3\theta} + \ldots\ldots,$$

and consider the operators

$$P = \lambda A_0\partial_{a_n} + (\lambda+\mu)A_1\partial_{a_{n+1}} + (\lambda+2\mu)A_2\partial_{a_{n+2}} + \ldots\ldots,$$

$$Q = \lambda' A_0'\partial_{a_{n'}} + (\lambda'+\mu')A_1'\partial_{a_{n'+1}} + (\lambda'+2\mu')A_2'\partial_{a_{n'+2}} + \ldots\ldots$$

Regarding e^θ as an operative symbol defined by the equation

$$e^{\kappa\theta}[\partial_{a_0}] = \partial_{a_\kappa},$$

we may write

$$\begin{aligned} P &= \{\lambda A_0e^{n\theta} + (\lambda+\mu)A_1e^{(n+1)\theta} + (\lambda+2\mu)A_2e^{(n+2)\theta} + \ldots\ldots\}[\partial_{a_0}] \\ &= e^{n\theta}\lambda(A_0 + A_1e^\theta + A_2e^{2\theta} + \ldots\ldots)[\partial_{a_0}] \\ &\qquad + e^{n\theta}\mu(A_1e^\theta + 2A_2e^{2\theta} + \ldots\ldots)[\partial_{a_0}] \\ &= e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)[\partial_{a_0}]. \end{aligned}$$

Similarly,

$$Q = e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)\psi(u)[\partial_{a_0}].$$

Now,

$$PQ - QP = \left\{Pe^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)\psi(u) - Qe^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right\}[\partial_{a_0}]$$

$$= \left\{e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)P\psi(u) - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)Q\phi(u)\right\}[\partial_{a_0}].$$

For $$Q\phi(u) = QA_0 + e^{\theta}QA_1 + e^{2\theta}QA_2 + \ldots\ldots;$$

so that $$e^{n\theta}\frac{d}{d\theta}Q\phi(u) = e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots\ldots)$$

and $$e^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^{\theta}A_1 + 2e^{2\theta}A_2 + \ldots\ldots);$$

so that $$Qe^{n\theta}\frac{d}{d\theta}\phi(u) = e^{n\theta}(e^{\theta}QA_1 + 2e^{2\theta}QA_2 + \ldots\ldots)$$

$$= e^{n\theta}\frac{d}{d\theta}Q\phi(u).$$

Similarly, $$Pe^{n'\theta}\frac{d}{d\theta}\psi(u) = e^{n'\theta}\frac{d}{d\theta}P\psi(u).$$

Moreover,

$$P\psi(u) = \psi'(u)Pu = \psi'(u)P(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \ldots\ldots)$$

$$= \psi'(u)\{e^{n\theta}\lambda A_0 + e^{(n+1)\theta}(\lambda + \mu)A_1 + e^{(n+2)\theta}(\lambda + 2\mu)A_2 + \ldots\ldots\}$$

$$= e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u).$$

Similarly, $$Q\phi(u) = e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u).$$

Hence

$$PQ - QP = \left\{e^{n'\theta}\left(\lambda' + \mu'\frac{d}{d\theta}\right)e^{n\theta}\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$

$$\left. - e^{n\theta}\left(\lambda + \mu\frac{d}{d\theta}\right)e^{n'\theta}\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}]$$

$$= e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\psi'(u)\left(\lambda + \mu\frac{d}{d\theta}\right)\phi(u)\right.$$

$$\left. - \left(\lambda + \mu n' + \mu\frac{d}{d\theta}\right)\phi'(u)\left(\lambda' + \mu'\frac{d}{d\theta}\right)\psi(u)\right\}[\partial_{a_0}].$$

If in this we write

$$\phi = \frac{u^2}{2},\quad \lambda = 4,\ \mu = 1,\ n = 1,$$

$$\psi = \log u,\ \lambda' = 0,\ \mu' = 1,\ n' = -1,$$

we have

$$PQ - QP = \left\{\left(1+\frac{d}{d\theta}\right)u^{-1}\left(4+\frac{d}{d\theta}\right)\frac{u^2}{2} - \left(3+\frac{d}{d\theta}\right)u\frac{d}{d\theta}\log u\right\}[\partial_{a_0}]$$

$$= \left\{\left(1+\frac{d}{d\theta}\right)\left(2u+\frac{du}{d\theta}\right) - \left(3+\frac{d}{d\theta}\right)\frac{du}{d\theta}\right\}[\partial_{a_0}]$$

$$= \left\{\left(1+\frac{d}{d\theta}\right)\left(2+\frac{d}{d\theta}\right) - \left(3+\frac{d}{d\theta}\right)\frac{d}{d\theta}\right\}u\,[\partial_{a_0}]$$

$$= 2u\,[\partial_{a_0}].$$

Now,

$$2u\,[\partial_{a_0}] = 2\,(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \ldots .)[\partial_{a_0}]$$

$$= 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots .).$$

Also

$$P = 4A_0\partial_{a_1} + 5A_1\partial_{a_2} + 6A_2\partial_{a_3} + \ldots .\,,$$

$$Q = A_1'\partial_{a_0} + 2A_2'\partial_{a_1} + 3A_3'\partial_{a_2} + \ldots .\,,$$

where

$$\frac{1}{2}(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \ldots .)^2 = A_0 + A_1e^{\theta} + A_2e^{2\theta} + \ldots .$$

and

$$\log\,(a_0 + a_1e^{\theta} + a_2e^{2\theta} + \ldots .) = \log a_0 + A_1'e^{\theta} + A_2'e^{2\theta} + \ldots .$$

Equating coefficients, we have

$$A_0 = \frac{1}{2}a_0^2,\ A_1 = a_0a_1,\ A_2 = a_0a_2 + \frac{a_1^2}{2}, \ldots .$$

$$A_1' = \frac{a_1}{a_0},\ A_2' = \frac{2a_0a_2 - a_1^2}{2a_0^2}, \ldots .$$

It is easily seen by expanding the logarithm that the general value of A_n' is $(-)^{n+1}\frac{S_n}{n}$ where S_n denotes the sum of the n^{th} powers of the roots of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots . + a_n.$$

Thus we have shown that if

$$P = 2a_0^2\partial_{a_1} + 5a_0a_1\partial_{a_2} + (6a_0a_2 + 3a_1^2)\,\partial_{a_3}$$

and

$$Q = \frac{a_1}{a_0}\partial_{a_0} + \frac{2a_0a_2 - a_1^2}{a_0^2}\partial_{a_1} + \frac{3a_0^2a_3 - 3a_0a_1a_2 + a_1^3}{a_0^3}\partial_{a_2} + \ldots .\,,$$

then

$$PQ - QP = 2\,(a_0\partial_{a_0} + a_1\partial_{a_1} + a_2\partial_{a_2} + \ldots .) = 2i.$$

The general formula obtained for $PQ - QP$ is an extension of a result of Capt. MacMahon's, who considers the case in which

$$\phi\,(u) = \frac{u^m}{m},\ \psi\,(u) = \frac{u^{m'}}{m'}.$$

When $\phi\,(u)$ and $\psi\,(u)$ have these values, the general formula becomes

$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu'n + \mu'\frac{d}{d\theta}\right)\left(\frac{\lambda u^{m+m'-1}}{m} + \mu u^{m+m'-2}\frac{du}{d\theta}\right)\right.$$

$$\left. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\}[\partial_{a_0}].$$

But
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m}\, u^{m+m'-1} + \mu u^{m+m'-2}\frac{du}{d\theta}\right)$$
$$= \left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right) u^{m+m'-1}.$$
Consequently
$$PQ - QP = e^{(n+n')\theta}\left\{\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left(\frac{\lambda}{m} + \frac{\mu}{m+m'-1}\frac{d}{d\theta}\right) - \ldots\ldots\ldots\ldots\ldots\ldots\right\} u^{m+m'-1}[\partial_{a_0}].$$
In Capt. MacMahon's notation
$$P = (m, \lambda, \mu, n),\quad Q = (m', \lambda', \mu', n');$$
in our notation
$$P = e^{n\theta}\left(\lambda + \mu \frac{d}{d\theta}\right)\frac{u^m}{m}[\partial_{a_0}],$$
$$Q = e^{n'\theta}\left(\lambda' + \mu' \frac{d}{d\theta}\right)\frac{u^{m'}}{m'}[\partial_{a_0}].$$
If now we write
$$PQ - QP = e^{(n+n')\theta}\left(\lambda_1 + \mu_1 \frac{d}{d\theta}\right)\frac{u^{m+m'-1}}{m+m'-1}[\partial_{a_0}],$$
which is equivalent to
$$PQ - QP = (m + m' - 1, \lambda_1, \mu_1, n + n'),$$
we have
$$\left(\lambda' + \mu' n + \mu' \frac{d}{d\theta}\right)\left\{\frac{\lambda}{m}(m + m' - 1) + \mu\frac{d}{d\theta}\right\}$$
$$- \left(\lambda + \mu n' + \mu \frac{d}{d\theta}\right)\left\{\frac{\lambda'}{m'}(m + m' - 1) + \mu'\frac{d}{d\theta}\right\} = \lambda_1 + \mu_1\frac{d}{d\theta}.$$
Hence we obtain
$$\lambda_1 = (m + m' - 1)\left\{\frac{\lambda}{m}(\lambda' + \mu' n) - \frac{\lambda'}{m'}(\lambda + \mu n')\right\},$$
$$\mu_1 = \mu\mu'(n - n') + \frac{\lambda\mu'}{m}(m' - 1) - \frac{\lambda'\mu}{m'}(m - 1).$$
This agrees with Capt. MacMahon's result, a statement of which was given in Lecture XX.

Let Q be a reversor to the operator $P = \lambda a^m \partial_b + (\ldots.)\,\partial_c + (\ldots.)\,\partial_d + \ldots.,$ and suppose that
$$(PQ - QP)\,F = \varkappa a^{m-1}F,$$
where F is any homogeneous and isobaric function and $\varkappa$ some number depending on its degree and weight. Then $\lambda a Q - \varkappa b$ will be the generator corresponding to Q. In other words, we have to prove that
$$P(\lambda a Q - \varkappa b)\,F = 0 \text{ whenever } PF = 0.$$

Now, by hypothesis, $Pa = 0$, $Pb = \lambda a^m$, and when $PF = 0$,

$$PQF = \varkappa a^{m-1} F.$$

Thus,

$$\begin{aligned} P(\lambda a Q - \varkappa b) F &= \lambda a PQF - \varkappa F . Pb \\ &= \lambda \varkappa a^m F - \lambda \varkappa a^m F = 0. \end{aligned}$$

As an example, consider the case of the reversor $\frac{d}{dx}$ in the theory of reciprocants. Here

$$P = V,\ \lambda = 2,\ m = 2;$$

and since

$$\left(V \frac{d}{dx} - \frac{d}{dx} V\right) F = 2\mu a F,$$

we have $\varkappa = 2\mu$. Hence the corresponding generator is $2\left(a \frac{d}{dx} - \mu b\right)$; or, disregarding the numerical factor 2, we may take $a \frac{d}{dx} - \mu b$ for the generator in question, which is usually denoted by the letter G.

We may also write G in the equivalent form

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \dots,$$

which it is sometimes more convenient to use.

I shall now show that

$$\Omega G - G\Omega = aw - b\Omega,$$

where w is the weight of the operand.

It is very easily seen that

$$\begin{aligned} \Omega(ac - b^2) &= 0, \\ \Omega(ad - bc) &= 2(ac - b^2), \\ \Omega(ae - bd) &= 3(ad - bc), \\ \Omega(af - be) &= 4(ae - bd), \\ &\dots\dots\dots\dots \end{aligned}$$

Hence it follows, by a direct and very simple calculation, that

$$\Omega G - G\Omega = 2(ac - b^2)\partial_c + 3(ad - bc)\partial_d + 4(ae - bd)\partial_e + \dots$$

But, since

$$b\partial_b + 2c\partial_c + 3d\partial_d + 4e\partial_e + \dots = w,$$

and

$$a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + \dots = \Omega,$$

$$aw - b\Omega = 2(ac - b^2)\partial_c + 3(ad - bc)\partial_d + 4(ae - bd)\partial_e + \dots$$

Consequently

$$\Omega G - G\Omega = aw - b\Omega.$$

The use of this formula will be seen in a subsequent lecture.

We may also prove an analogous theorem relating to the invariant generator $a \frac{d}{dx} - \nu b$, which we shall call G'.

Let the operand be F, a homogeneous and isobaric function of degree i and weight w. Then VF is of degree $i+1$ and weight $w-1$; its ν is therefore

$$3(i+1)+2(w-1)=\nu+1.$$

Thus, $$(VG'-G'V)F=\left\{V\left(a\frac{d}{dx}-\nu b\right)-\left(a\frac{d}{dx}-\nu b-b\right)V\right\}F$$

$$=a\left(V\frac{d}{dx}-\frac{d}{dx}V\right)F-\nu(Vb-bV)F+bVF.$$

But $$\left(V\frac{d}{dx}-\frac{d}{dx}V\right)F=2\mu aF=2(3i+w)aF$$

and $$VbF=bVF+2a^2F.$$

Consequently

$$VG'-G'V=2(3i+w)a^2F-2\nu a^2F+bVF$$
$$=2(3i+w-\nu)a^2F+bVF$$
$$=-2wa^2F+bVF.$$

It is perhaps worthy of notice that if I is an invariant of weight w and R a pure reciprocant, also of weight w, then

$$\Omega GI=awI \text{ and } VG'R=-2a^2wR;$$

whereas $$\Omega G'I=0 \quad \text{ and } VGR=0.$$

LECTURE XXVII.

I should like to make a momentary pause in the development of the theory which now engages our attention and to revert to the proof of Cayley's theorem for the enumeration of linearly independent invariants contained in Lecture XI and expressed by the formula $(w;\ i,\ j)-(w-1;\ i,\ j)$.

Since that proof was written out I have endeavored to obtain one that might be capable of being extended to the supposed analogous theorem, regarding pure reciprocants, expressed by the formula $(w;\ i, j)-(w-1;\ i+1,\ j)$, but all my efforts and those of another and most skilful algebraist in this direction have hitherto proved ineffectual.

In aiming at this object, however, I obtained a second proof of Cayley's theorem less compendious than the previous one, and subject to the drawback that it assumes the law of Reciprocity, but which possesses the advantage over it of being more direct and looking the question, so to say, more squarely in the face. The forms of thought employed in it seem to me too peculiar and precious

to be consigned to oblivion. I am not one of those who look upon Analysis as only valuable for the positive results to which it leads, and who regard proofs as almost a superfluity, thinking it sufficient that mathematical formulae should be obtained, no matter how, and duly entered on a register.

I look upon Mathematics not merely as a language, an art, and a science, but also as a branch of Philosophy, and regard the forms of reasoning which it embodies and enshrines as among the most valuable possessions of the human mind. Add to this that it is scarcely possible that a well-reasoned mathematical proof shall not contain within itself subordinate theorems—germs of thought of intrinsic value and capable of extended application.

That such was the opinion of our High Pontiff is shown by the publication of his seven proofs of the Theorem of Reciprocity, a number to which subsequent researches have made almost annual additions (like so many continually augmenting asteroids in the Arithmetical Firmament) to such an extent that it would seem to be an interesting task for some one to undertake to form a corolla of these various proofs and to construct a reasoned bibliography, a *catalogue raisonnée*, of this one single theorem. For these reasons, I shall venture to put on record (*valeat quantum*) the following Second Proof of Cayley's Theorem.

The notation which I proceed to explain will be found very convenient. A rational integral homogeneous isobaric function will be called a *gradient;* its weight, degree, extent (extent meaning the number of letters after the first) will be denoted by w; i, j and spoken of as the *type* of the gradient. Either a single letter, such as ϕ, will be employed to denote a gradient, or else its type enclosed in a parenthesis thus $[w;\ i,\ j]$. The abbreviation $T\phi$ signifies the type of ϕ; thus, $T\phi = w;\ i,\ j$.

The number of terms in the most general gradient whose type is the same as that of ϕ will be spoken of as the *denumerant* of ϕ. The letter N will be used to denote such a denumerant; thus, $N\phi$ signifies the denumerant of ϕ.

In like manner, the letter Δ will be used to denote the number of linear relations between the coefficients of any gradient, whenever such relations exist. Hence $N\phi - \Delta\phi$ expresses the number of terms in ϕ whose coefficients are left arbitrary. Obviously, when ϕ is the most general gradient of its type, we have

$$\Delta\phi = 0.$$

We also use E to denote the $ij - 2w$, which may be called the *excess*, of the gradient of type $w;\ i,\ j$. Thus, if $T\phi = w;\ i,\ j$, we write $E\phi = ij - 2w$.

The operators which we shall employ, viz. Ω and Ω', are defined by the equations

$$\Omega = a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots\ldots,$$
$$\Omega' = \quad a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots\ldots$$

The first of these is of course an equivalent, but for present purposes more convenient, form of $a\partial_b + 2b\partial_c + 3c\partial_d + \ldots\ldots$, the ordinary invariant annihilator Ω $\left(\text{as will be evident on writing } a_0 = a,\ a_1 = \frac{b}{1},\ a_2 = \frac{c}{1.2}, \ldots\right)$; the second of them, Ω', is merely Ω deprived of its first term.

We may now give the following enunciation of the theorem to be proved: *If ϕ is the most general gradient of its type, $\Omega\phi$ is also the most general gradient of its type whenever $E\phi$ is not negative.* In other words, we shall prove that, subject to the condition stated above, $\Delta\Omega\phi = 0$ whenever $\Delta\phi = 0$. This is equivalent to Cayley's Theorem on the number of linearly independent invariants. For the number of forms of the same type as ϕ, and subject to annihilation by Ω, is
$$N\phi - N\Omega\phi + \Delta\Omega\phi;$$
and Cayley's Theorem states that the number of such forms is $N\phi - N\Omega\phi$, which will be the case when
$$\Delta\Omega\phi = 0.$$

The theorem of Reciprocity enables us to dispense with the discussion of those cases in which the extent j is greater than the degree i. For since (see *American Journal of Mathematics*, Vol. I, p. 91) the number of linearly independent invariants for the type $w;\, j,\, i$ is the same as for the type $w;\, i,\, j$, we can substitute the first of these types for the second, using ψ, whose type is $w;\, j,\, i$, instead of ϕ, whose type is $w;\, i,\, j$. Thus we have
$$N\psi - N\Omega\psi + \Delta\Omega\psi = N\phi - N\Omega\phi + \Delta\Omega\phi.$$
But by Ferrers' proof of Euler's Theorem (*vide* A Constructive Theory of Partitions, Vol. V, No. 3 of this Journal),
$$N\psi = N\phi \text{ and } N\Omega\psi = N\Omega\phi.$$
It obviously follows that
$$\Delta\Omega\psi = \Delta\Omega\phi.$$
Cases for which the extent is greater than the degree may therefore be made to depend on those for which the degree is greater than the extent. Hence Cayley's Theorem depends on the proof that $\Delta\Omega\phi = 0$ when $i => j$ and $ij => 2w$.

In the course of the demonstration, the following Lemma will be used:
If $\quad T\phi = w;\, i,\, j$ and $T\psi = ij - w;\, i,\, j$, then $N\phi = N\psi$.

The types of the two gradients we are now considering may be said to be *complementary*, and then the Lemma may be enunciated in words as follows:

The denumerants of two gradients are equal when the types of the gradients are complementary.

The proof consists in showing that to each term of the type $w;\ i,\ j$ there corresponds a term of the type $ij - w;\ i,\ j$. Let $a_0^{\lambda_0} a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_j^{\lambda_j}$ be any term of the type $w;\ i,\ j$; then

$$w = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j$$

and

$$i = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_j.$$

Writing the suffixes of the letters $a_0,\ a_1,\ a_2, \ldots a_j$ in reverse order, everything else being kept unchanged, we obtain the term $a_j^{\lambda_0} a_{j-1}^{\lambda_1} a_{j-2}^{\lambda_2} \ldots a_0^{\lambda_j}$, whose weight we will call w'. Then

$$\begin{aligned} w' &= j\lambda_0 + (j-1)\lambda_1 + (j-2)\lambda_2 + \ldots + \lambda_{j-1} \\ &= j(\lambda_0 + \lambda_1 + \lambda_2 + \ldots + \lambda_j) - (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \ldots + j\lambda_j) \\ &= ij - w. \end{aligned}$$

The degree of the transformed term is still i, and its extent is still j, while its weight has become $ij - w$; its type is therefore complementary to that of the original term. Hence to each term of any given type there corresponds a term of the complementary type, and consequently the total number of possible terms (*i. e.* the Denumerant) for each type is the same.

By means of this Lemma it can be shown that $\Delta\Omega\phi = 0$ when $E\phi = -1$.

Let $$T\phi = w;\ i,\ j \text{ where } ij - 2w = -1;$$

then, since $T\Omega\phi = w = 1;\ i,\ j$, the types $T\phi$ and $T\Omega\phi$ are complementary (the sum of the weights being $w + w - 1 = ij$).

It follows from the Lemma that the Denumerants of ϕ and $\Omega\phi$ are equal.

Hence $$\Delta\Omega\phi = 0.$$

For if not, the number of independent terms in $\Omega\phi$ being less than the denumerant of $\Omega\phi$, will also be less than its equal, the denumerant of ϕ, and therefore there will be one or more invariants of the type $w;\ i,\ j$ for which the excess is negative. Since this is known to be impossible, we must have

$$\Delta\Omega\phi = 0.$$

We next prove that, in all cases for which $i \mathrel{=\!\!>} w$, the number of linearly independent invariants of the type $w;\ i,\ j$ is correctly given by the formula

$$(w;\ i,\ j) - (w - 1;\ i,\ j),$$

which is equivalent (as we showed at the beginning of Lecture XV) to

$$(w;\, w,\, j) - (w-1;\, w,\, j),$$

or, what is the same thing, to the coefficient of $a^w x^w$ in the expansion of

$$F = \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)\ldots\ldots(1-ax^j)}\,.$$

Let the expansion of

$$G = \frac{1-x}{(1-ax)(1-ax^2)(1-ax^3)\ldots\ldots(1-ax^j)}$$

be $$1 + (a-1)x + A_2x^2 + \ldots\ldots + A_w x^w + \ldots\ldots$$

The expansion of F is obtained by multiplying that of G by the infinite geometrical series $$1 + a + a^2 + a^3 + \ldots\ldots$$

But we only require the coefficient of $a^w x^w$ in the expansion of F, so that we need only retain the portion

$$A_w x^w (1 + a + a^2 + \ldots\ldots + a^w)$$

of the above product instead of its complete expression.

It is of importance to notice here that A_w, which is independent of x, cannot contain any higher power of a than a^w. (That this is so will be evident from the constitution of the fraction G, for clearly no power of a in the expansion of G can be associated with a lower power of x.) Thus we see that

$$A_w = \alpha a^w + \beta a^{w-1} + \gamma a^{w-2} + \ldots\ldots + \kappa a + \lambda,$$

and consequently

$$\begin{aligned} &A_w x^w (1 + a + a^2 + \ldots\ldots + a^w) \\ &= \ldots\ldots + a^w x^w (\alpha + \beta + \gamma \ldots\ldots + \kappa + \lambda) + \ldots\ldots \end{aligned}$$

Hence the coefficient of $a^w x^w$ in the expansion of F is

$$\alpha + \beta + \gamma + \ldots\ldots + \kappa + \lambda,$$

which is the value assumed by A_w when in it we write $a = 1$. Call this value A'_w, and let the value of G when $a = 1$ be denoted by G'. Then A'_w is the coefficient of x^w in

$$G' = \frac{1}{(1-x^2)(1-x^3)\ldots\ldots(1-x^j)}\,.$$

Hence we see that, when $i \equiv> w$, the value of $(w;\, i,\, j) - (w-1;\, i,\, j)$ is the total number of ways in which w can be made up of the parts $2, 3, \ldots\ldots j$.

We have yet to show that this number is the same as that of the linearly independent invariants of the type $w;\, i,\, j$ when $i \equiv> w$.

This follows from the known theorem that every invariant is either a rational integral function of the Protomorphs a, P_2, P_3, $\ldots\ldots$ P_j (meaning the

invariant a and those of the second and third degrees alternately whose first terms are ac, a^2d, ae, a^2f, ...), or can be made so by multiplying it by a suitable power of a. Thus, if I be any invariant of degree i and weight w,

$$Ia^{w-i} = \Phi(a,\ P_2,\ P_3,\ \ldots\ P_j),$$

where Φ, which is of degree-weight $w.w$ when expressed in terms of $a, b, c, \ldots,$ is rational and integral as regards the protomorphs.

When $i => w$, writing

$$I = a^{i-w}\Phi(a,\ P_2,\ P_3,\ \ldots\ P_j),$$

Φ consists of a series of terms of the form $Aa^\theta P_2^\lambda P_3^\mu \ldots P_j^\rho$, *each with an arbitrary coefficient*, where, since

$$2\lambda + 3\mu + 4\nu + \ldots + j\rho = w,$$

the number of arbitrary constants in Φ is the total number of partitions of w into parts $2, 3, \ldots j$. Hence the number of linearly independent invariants of the type $w;\ i, j$ is also this number of partitions, *i. e.* by what precedes is $(w;\ i,\ j) - (w-1;\ i,\ j)$. This proves Cayley's theorem for cases in which $i => w$.

But when $i < w$, the equation

$$Ia^{w-i} = \Phi(a,\ P_2,\ P_3,\ \ldots\ P_j)$$

shows that the coefficients of Φ are not all arbitrary, but must be so chosen that Φ may be divisible by a^{w-i}, and the reasoning employed in the case of $i => w$ no longer holds.

It will be convenient at this point of the investigation to review the results we have hitherto obtained and to see what remains to be proved.

Cayley's Theorem has been demonstrated for cases in which the degree is not less than the weight. This will be expressed by saying that

$$\Delta\Omega\,[w;\ i, j] = 0 \ \text{ when } \ i => w.$$

We have also proved that

$$\Delta\Omega\,[w;\ i, j] = 0 \ \text{ when } \ ij - 2w = -1.$$

The law of reciprocity has been expressed in the form

$$\Delta\Omega\,[w;\ i, j] = \Delta\Omega\,[w;\ j,\ i],$$

where $[w;\ i, j]$ denotes the most general gradient of the type $w;\ i, j$.

The theorem to be proved is that

$$\Delta\Omega\,[w;\ i, j] = 0 \ \text{ when } \ ij - 2w => 0;$$

but we may at once dismiss those cases in which $i => w$, and (assuming the theorem to have been proved for Quantics of order inferior to j) those in which $i < j$, for these depend on the truth of the theorem for a Quantic of order i.

It remains, then, to prove that, when $ij - 2w => 0$, $\Delta\Omega[w; i, j] = 0$ for values of i inferior to w, but not inferior to j. This may be effected as follows: Let ϕ be the most general gradient of the type $w; i+1, j$, and suppose

$$\phi = P + Qa + Ra^2 + Sa^3,$$

where P, Q and R do not contain the letter a, though S may do so. Then, writing

$$\phi_1 = Q + Ra + Sa^2,$$

ϕ_1 is the most general gradient of the type $w; i, j$.

Now, if $\Omega = a\partial_b + b\partial_c + c\partial_d + \ldots.$, and $\Omega' = b\partial_c + c\partial_d + \ldots.$, we have

$$(1) \qquad \Omega\phi = \Omega' P + \left(\Omega' Q + \frac{dP}{db}\right)a + \left(\Omega' R + \frac{dQ}{db}\right)a^2 + \left(\Omega S + \frac{dR}{db}\right)a^3$$

and $$\Omega\phi_1 = \Omega' Q + \left(\Omega' R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2.$$

Confining our attention for the present to $\Omega\phi_1$, it is clear that if no linear relations exist among the coefficients of $\Omega' R$ (*i. e.* if $\Delta\Omega' R = 0$) the coefficients of $\Omega' Q$ are not connected with those of $\Omega' R + \frac{dQ}{db}$ by any linear relation. For the coefficient of each term of $\Omega' R + \frac{dQ}{db}$ is the sum of a single coefficient of Q and an independent linear function of the coefficients of R. Moreover, obviously the coefficients of $\Omega' Q$ are unconnected with those of $\Omega S + \frac{dR}{db}$.

If, then, the coefficients of $\Omega' Q$ are not related *inter se* (*i. e.* if $\Delta\Omega' Q = 0$), we have

$$(2) \qquad \Delta\Omega\phi_1 = \Delta\left\{\left(\Omega' R + \frac{dQ}{db}\right)a + \left(\Omega S + \frac{dR}{db}\right)a^2\right\}.$$

Looking now to the expression (1) for $\Omega\phi$, we see immediately from (2) that any linear relation subsisting between the coefficients of $\Omega\phi_1$ will also subsist between those of $\Omega\phi$, and therefore that $\Delta\Omega\phi_1$ is not greater than $\Delta\Omega\phi$.

If, then, $\Delta\Omega\phi = 0$, it follows that $\Delta\Omega\phi_1 = 0$, provided that both the supplementary conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are also satisfied.

Now, since $\phi_1 = Q + Ra + Sa^2$ is the most general gradient of the type $w; i, j$,

Q will be the most general gradient of the type $w - i; i, j - 1$

and

R " " " " " " " " " $w - i + 1; i - 1, j - 1$,

when in Q and R we change $b, c, d, \ldots$ into $a, b, c, \ldots$. This change converts $\Omega' = b\partial_c + c\partial_d + \ldots$ into $\Omega = a\partial_b + b\partial_c + \ldots$. Hence the conditions $\Delta\Omega' Q = 0$ and $\Delta\Omega' R = 0$ are respectively equivalent to

$$\Delta\Omega\,[w - i;\, i, j - 1] = 0 \text{ and } \Delta\Omega\,[w - i + 1;\, i - 1, j - 1] = 0.$$

Supposing these supplementary conditions to be satisfied, what we have proved is that when

$$\Delta\Omega\,[w;\, i + 1, j] = 0 \ (i.\,e.\ \Delta\Omega\phi = 0),$$

then also

$$\Delta\Omega\,[w;\, i, j] \quad = 0 \ (i.\,e.\ \Delta\Omega\phi_1 = 0).$$

Now,

$$\begin{aligned} T\phi &= w;\, i + 1, j, && \text{so that } E\phi = (i + 1)j - 2w = (ij - 2w) + j, \\ TQ &= w - i;\, i, j - 1, && \text{“ \ “ } \ EQ = i(j - 1) - 2(w - i) = (ij - 2w) + i, \\ TR &= w - i + 1;\, i - 1, j - 1, && \text{so that } ER = (i - 1)(j - 1) - 2(w - i + 1) \\ & && \qquad\qquad = (ij - 2w) + i - j - 1. \end{aligned}$$

Thus, when $ij - 2w => 0$ and $i => j$,

$E\phi$ and EQ are both positive.

ER is in general $=> 0$, but in the special case where $ij - 2w = 0$ and $i = j$, we have $ER = -1$. Except in this case (which gives us no trouble, since we have seen that $\Delta\Omega R = 0$ in consequence of $ER = -1$), we have never to deal with a type of which the excess is negative.

Hence, if we assume Cayley's Theorem to have been proved for all extents up to $j - 1$ inclusive, we have

$$\Delta\Omega\,[w - i;\ i, j - 1] = 0$$

and

$$\Delta\Omega\,[w - i + 1;\, i - 1, j - 1] = 0$$

($i.\,e.$ the two supplementary conditions are satisfied).

We wish to extend the theorem to the extent j.

Subject to the conditions $i => j$ and $ij - 2w => 0$, we have

$$\Delta\Omega\,[w;\, i, j] = 0 \text{ if } \Delta\Omega\,[w;\, i + 1, j] = 0.$$

But we need consider no value of i greater than w, as we have proved that

$$\Delta\Omega\,[w;\, w, j] = 0 = \Delta\Omega\,[w;\, w + x, j];$$

therefore

$$\begin{aligned} \Delta\Omega\,[w;\, w - 1, j] &= 0, \\ \Delta\Omega\,[w;\, w - 2, j] &= 0, \\ \cdots\cdots\cdots & \\ \Delta\Omega\,[w;\, j, j] \quad &= 0. \end{aligned}$$

As previously shown, the theorem is true for all values of i inferior to j if it is true for all Quantics of inferior order. Thus the theorem is true for a Quantic of order j and for every value of i if it is true for all Quantics of order

inferior to j. But it is true for the Quadric (where $j=2$);* therefore also for the Cubic ($j=3$); therefore also for the Quartic ($j=4$), and so universally. Hence the theorem to be proved is demonstrated.

LECTURE XXVIII.

We now resume the theory of Principiants and proceed to prove the important theorem that every Principiant is either simply an invariant in respect to a known series of pure reciprocants, which we call A, B, C, D, ..., or else becomes such an invariant when multiplied by a^{w-i}, where w is the weight and i the degree of the Principiant in question. Using the letter M to denote the pure reciprocant $ac - \frac{5}{4}b^2$, and G the ordinary eductive generator,

$$4(ac-b^2)\partial_b + 5(ad-bc)\partial_c + 6(ae-bd)\partial_d + 7(af-be)\partial_e + \ldots .$$

(which, it will be remembered, is only another form of $a\frac{d}{dx} - \mu b$, with the advantage of the μ being suppressed, *i. e.* only implicitly contained), we obtain in succession the values of A, B, C, D, ... from the following equations:

$$\begin{aligned}
5A &= GM,\\
6B &= GA,\\
7C &= GB - MA,\\
8D &= GC - 2MB,\\
9E &= GD - 3MC,\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

On performing the calculations indicated by these equations we shall find

$$\begin{aligned}
A &= a^2d - 3abc + 2b^3,\\
B &= a^3e - 2a^2c^2 - \frac{7}{2}a^2bd + \frac{17}{2}ab^2c - 4b^4,\\
C &= a^4f - 5a^3cd - 4a^3be + 13a^2bc^2 + \frac{45}{4}a^2b^2d - \frac{103}{4}ab^3c + \frac{19}{2}b^5,\\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3 + \text{terms involving } b,\\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d + \text{terms involving } b.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

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* When $j=2$ the condition $ij => 2w$ becomes identical with $i => w$; but we have already seen that the theorem is true whenever $i => w$.

The fact that D is a pure reciprocant enables us to calculate the terms in E which are independent of b without a previous knowledge of the values of those terms in D which involve b. For, since

$$G = 4(ac - b^2)\partial_b + \ldots \text{ and } V = 2a^2\partial_b + \ldots,$$

$$a^2G - 2(ac - b^2)V \text{ does not contain } \partial_b.$$

Hence the operation of $a^2G - 2(ac - b^2)V$ on terms involving b cannot give rise to terms independent of b. But,

$$D \text{ being a pure reciprocant, } VD = 0\,;$$

so that

$$\{a^2G - 2(ac - b^2)V\}D = a^2GD,$$

and the terms of a^2GD which do not involve b are found by operating with

$$[a^2G - 2(ac - b^2)V]_{b=0}$$

on the terms of D which do not involve b.

If, now, we use $M_0, A_0, B_0, C_0, \ldots$ to denote those portions of $M, A, B, C, \ldots$ which are independent of b, and write

$$[a^2G - 2(ac - b^2)V]_{b=0} = a^2G_0,$$

we shall still have

$$9E_0 = G_0D_0 - 3M_0C_0\,;$$

and in general the law of successive derivation for $A_0, B_0, C_0, D_0, \ldots$ is the same as that for $A, B, C, D, \ldots$ except that G_0 takes the place of G.

We have

$$\begin{aligned} a^2G_0 &= [a^2G - 2(ac - b^2)V]_{b=0} \\ &= a^2(5ad\partial_c + 6ae\partial_d + 7af\partial_e + 8ag\partial_f + 9ah\partial_g + \ldots) \\ &\quad - 2ac\{6ac\partial_d + 7ad\partial_e + (8ae + 4c^2)\partial_f + (9af + 9cd)\partial_g + \ldots\}; \end{aligned}$$

so that

$$\begin{aligned} G_0 = {} & 5ad\partial_c + 6(ae - 2c^2)\partial_d + 7(af - 2cd)\partial_e \\ & + \frac{8}{a}(a^2g - 2ace - c^3)\partial_f + \frac{9}{a}(a^2h - 2acf - 2c^2d)\partial_g + \ldots; \end{aligned}$$

and consequently (since $M_0 = ac$),

$$\begin{array}{lll} 5A_0 = G_0M_0 & \text{gives} & A_0 = a^2d, \\ 6B_0 = G_0A_0 & \text{"} & B_0 = a^3e - 2a^2c^2, \\ 7C_0 = G_0B_0 - M_0A_0 & \text{"} & C_0 = a^4f - 5a^3cd, \\ 8D_0 = G_0C_0 - 2M_0B_0 & \text{"} & D_0 = a^5g - \dfrac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3, \\ 9E_0 = G_0D_0 - 3M_0C_0 & \text{"} & E_0 = a^6h - \dfrac{15}{2}a^5de - 7a^5cf + 29a^4c^2d, \\ \ldots\ldots\ldots & \text{"} & \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Thus, *ex. gr.*

$$8D_0 = G_0(a^4f - 5a^3cd) - 2ac(a^3e - 2a^2c^2)$$

$$= -25a^4d^2 - 30a^3c(ae - 2c^2) + 8a^3(a^2g - 2ace - c^3) - 2ac(a^3e - 2a^2c^2);$$

whence $$D_0 = a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3.$$

Again, $$9E_0 = G_0(a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3) - 3ac(a^4f - 5a^3cd)$$

$$= 5ad(-6a^4e + 21a^3c^2) - \frac{75}{2}(ae - 2c^2)a^4d - 42(af - 2cd)a^4c$$

$$+ 9(a^2h - 2acf - 2c^2d)a^4 - 3ac(a^4f - 5a^3cd)$$

gives $$E_0 = a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d.$$

Similarly, from the known values of D_0 and E_0 we may deduce that of the next letter, F_0, and so on to any extent.

It may be noticed that each of the pure reciprocants $A, B, C, D, \ldots$ can be determined without ambiguity, by means of the annihilator V, when the portions of them, $A_0, B_0, C_0, D_0, \ldots$ independent of b are known.

For suppose R and R' to be two reciprocants, of weight w, for each of which the terms independent of b are the same. Then their difference is divisible by b. Let

$$R - R' = b\phi;\ \text{then}\ V(b\phi) = 0;\ i.\,e.\ 2a^2\phi + bV\phi = 0.$$

Hence ϕ is divisible by b, and $R - R'$ is divisible by b^2; say $R - R' = b^2\psi$. Then

$$V(b^2\psi) = 4a^2b\psi + b^2V\psi = 0,$$

showing that ψ is divisible by b, and $R - R'$ by b^3.

By continually reasoning in this manner, we prove that $R - R'$ must be divisible by b^w; and then the remaining factor (being of weight 0) is necessarily of the form λa^θ, where λ and θ are numerical constants. Thus

$$R - R' = \lambda a^\theta b^w,\ \text{and consequently}\ V(\lambda a^\theta b^w) = 0.$$

This is impossible unless $\lambda = 0$, when the two reciprocants R, R' become equal, showing that there cannot be two different reciprocants for which the terms independent of b are the same. When, therefore, the terms which do not involve b of any pure reciprocant are known, the complete expression of that reciprocant can be determined without ambiguity.

Each reciprocant of the series $A, B, C, D, \ldots$ possesses the property of being, so to say, an Invariant relative to the one which precedes it, meaning that the operation of $\Omega = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$ on any letter gives (to a

factor près) the one immediately preceding it. The first letter, A, is an Invariant in the ordinary sense. We can in fact show that

$$\Omega A = 0,$$
$$\Omega B = A \times \frac{a}{2},$$
$$\Omega C = 2B \times \frac{a}{2},$$
$$\Omega D = 3C \times \frac{a}{2},$$
$$\Omega E = 4D \times \frac{a}{2},$$
$$\cdots\cdots\cdots\cdots$$

The proof depends on a formula established in Lecture XXVI of this course, viz.

$$\Omega G - G\Omega = wa - b\Omega,$$

where G is the generator $4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots.$, and w is the weight of the operand.

Thus, observing that the weights of A, B, C, D, ... are 3, 4, 5, 6, ... respectively, we have

$$(\Omega G - G\Omega)A = (3a - b\Omega)A,$$
$$(\Omega G - G\Omega)B = (4a - b\Omega)B,$$
$$(\Omega G - G\Omega)C = (5a - b\Omega)C,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Now, since A is the well-known invariant $a^2d - 3abc + 2b^3$, we may write $\Omega A = 0$ in the first of these equations, which then reduces to

$$\Omega GA = 3aA.$$

But, since
$$6B = GA,$$
we have
$$6\Omega B = \Omega GA = 3aA.$$
Thus
$$\Omega B = A \times \frac{a}{2}.$$

Again, substituting for ΩB in the formula

$$(\Omega G - G\Omega)B = (4a - b\Omega)B,$$

we find
$$\Omega GB - G\left(\frac{aA}{2}\right) = 4aB - \frac{ab}{2}A,$$

where, since G (which is linear in ∂_b, ∂_c, ... and does not contain ∂_a) does not operate on a,

$$G\left(\frac{aA}{2}\right) = \frac{a}{2}GA = 3aB,$$

and consequently
$$\Omega GB + \frac{ab}{2}A = 7aB.$$

Now, $\qquad 7C = GB - MA;$

so that $\qquad 7\Omega C = \Omega GB - A\Omega M - M\Omega A.$

But, since $\qquad \Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}$ and $\Omega A = 0$,

$$7\Omega C = \Omega GB + \frac{ab}{2} A = 7aB.$$

Thus $\qquad \Omega C = 2B \times \frac{a}{2}.$

We may, in exactly the same way, prove that

$$\Omega D = 3C \times \frac{a}{2},$$

$$\Omega E = 4D \times \frac{a}{2},$$

and so on to any extent.

In the following inductive proof it will be convenient to denote the letters

$$A,\ B,\ C,\ D,\ E, \ldots .$$

by

$$u_0,\ u_1,\ u_2,\ u_3,\ u_4, \ldots ,$$

and then the theorem to be proved is that

$$\Omega u_n = n u_{n-1} \times \frac{a}{2}.$$

When this notation is used, the law of successive derivation which defines the capital letters is expressed by the equation

(1) $\qquad (n+7)\,u_{n+2} - Gu_{n+1} + (n+1)\,Mu_n = 0,$

where G is the generator $4\,(ac - b^2)\,\partial_b + 5\,(ad - bc)\,\partial_c + \ldots .$, and $M = ac - \frac{5b^2}{4}$

Operating with Ω on the above equation, we obtain

(2) $\qquad (n+7)\,\Omega u_{n+2} - \Omega G u_{n+1} + (n+1)(M\Omega u_n + u_n \Omega M) = 0.$

Now, the weights of $u_0, u_1, u_2, \ldots .$ are $3, 4, 5, \ldots .$ respectively, and consequently the operation of

$$\Omega G - G\Omega \doteq wa - b\Omega$$

on u_{n+1} (whose weight is $n+4$) gives

$$(\Omega G - G\Omega)\,u_{n+1} = (n+4)\,au_{n+1} - b\Omega u_{n+1}.$$

Or, assuming that $\Omega u_\kappa = \kappa u_{\kappa-1} \times \frac{a}{2}$ for all values of κ as far as $n+1$ inclusive (it has previously been shown that $\Omega B = A \times \frac{a}{2}$ and $\Omega C = 2B \times \frac{a}{2}$, so that the theorem is true for $\kappa = 1$ and $\kappa = 2$),

$$\begin{aligned} \Omega G u_{n+1} &= G\Omega u_{n+1} + (n+4)\,au_{n+1} - b\Omega u_{n+1} \\ &= (n+1)\,G\left(\frac{a}{2}\,u_n\right) + (n+4)\,au_{n+1} - (n+1)\,\frac{ab}{2}\,u_n . \end{aligned}$$

But $\left(\text{remembering that } G \text{ does not operate on } a, \text{ so that } G.\frac{a}{2}u_n = \frac{a}{2}Gu_n\right)$ we have, in virtue of equation (1),

$$G\left(\frac{a}{2}u_n\right) = \frac{a}{2}\left\{(n+6)\,u_{n+1} + nMu_{n-1}\right\}.$$

Hence it follows that

$$\Omega G u_{n+1} = \frac{n+1}{2}a\{(n+6)\,u_{n+1} + nMu_{n-1}\} + (n+4)\,au_{n+1} - (n+1)\,\frac{ab}{2}u_n$$
$$= \frac{(n+2)(n+7)}{2}au_{n+1} + \frac{n(n+1)}{2}aMu_{n-1} - (n+1)\,\frac{ab}{2}u_n.$$

On substituting this in (2) we obtain

$$(n+7)\left\{\Omega u_{n+2} - (n+2)\frac{a}{2}u_{n+1}\right\}$$
$$+ (n+1)\,M\left\{\Omega u_n - n\frac{a}{2}u_{n-1}\right\}$$
$$+ (n+1)\,u_n\left\{\Omega M + \frac{ab}{2}\right\} = 0.$$

This reduces to

$$\Omega u_{n+2} = (n+2)\,\frac{a}{2}u_{n+1}.$$

For, according to the assumption previously made in the course of the demonstration,

$$\Omega u_n = n\frac{a}{2}u_{n-1};$$

so that the second term vanishes; and the third term vanishes because

$$\Omega M = \Omega\left(ac - \frac{5b^2}{4}\right) = -\frac{ab}{2}.$$

We have therefore proved that if the theorem is true for Ωu_κ, when κ has any value up to $n+1$ inclusive, it is also true for Ωu_{n+2}. But the theorem holds for $\kappa = 1$, and for $\kappa = 2$. It therefore holds universally for any positive integer value of κ.

Recalling the known values of the reciprocants $M, A, B, C, D, \ldots.$ we observe that their principal terms are $ac, a^2d, a^3e, a^4f, a^5g, \ldots.$, where it is to be noticed that the most advanced of the small letters in the expression for any capital letter occurs only in the first degree multiplied by a power of a. In other words, $M, A, B, C, D, \ldots.$ form a series of Protomorphs, and consequently every Pure Reciprocant can, as we have already seen (vide *American Journal of Mathematics*, Vol. IX, p. 35), be expressed as a function of $a, M, A, B, C, D, \ldots.$ rational in all of them and integral in all except a.

But it is further to be noticed that whereas

a is of	degree	1	and	weight	0,
M " "	"	2	"	"	2,
A " "	"	3	"	"	3,
B " "	"	4	"	"	4,

and in fact that every capital letter is of equal weight and degree.

From this it will follow that every Pure Reciprocant will be the product of a power of a into a function of the capital letters alone.

For let i be the degree and w the weight of any pure reciprocant expressed in terms of a, M, A, B, C, ... , and suppose one of its terms to be

$$a^\eta M^\theta A^\kappa B^\lambda C^\mu \ldots\ldots;$$

then
$$\eta + 2\theta + 3\kappa + 4\lambda + 5\mu + \ldots\ldots = i$$
and
$$2\theta + 3\kappa + 4\lambda + 5\mu + \ldots\ldots = w.$$
Hence
$$\eta = i - w,$$
which is the same for every term of the pure reciprocant in question. Thus each term contains a^{i-w} as a factor, and the reciprocant is of the form

$$a^{i-w}\Phi(M, A, B, C, D, \ldots\ldots).$$

Let us now consider any Principiant P; since P is a pure reciprocant, we must have
$$P = a^{i-w}\Phi(M, A, B, C, D, \ldots\ldots).$$
But Principiants are subject to annihilation by Ω, and consequently $\Omega P = 0$, which gives

$$\frac{d\Phi}{dM}\Omega M + \frac{d\Phi}{dA}\Omega A + \frac{d\Phi}{dB}\Omega B + \frac{d\Phi}{dC}\Omega C + \ldots\ldots = 0.$$

On writing for $\quad \Omega M \;,\; \Omega A ,\; \Omega B \;,\; \Omega C \;, \ldots\ldots$

their values $\quad -b \times \frac{a}{2},\; 0 ,\; A \times \frac{a}{2},\; 2B \times \frac{a}{2}, \ldots\ldots$

we obtain

$$\frac{a}{2}(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D + \ldots\ldots)\Phi = 0.$$

From this it would follow that Φ is an invariant in the two sets of letters

$$-b,\ M \text{ and } A, B, C, D, \ldots\ldots;$$

but it is easy to see that it is an invariant in the latter set exclusively. For M and A, B, C, D, ... being all of them pure reciprocants,

$$\Phi \text{ and } \partial_M\Phi,\ \partial_B\Phi,\ \partial_C\Phi,\ \partial_D\Phi, \ldots\ldots,$$

which are functions of M, A, B, C, ... exclusively, must also be pure reciprocants.

If, then, we operate with V on

$$(-b\partial_M + A\partial_B + 2B\partial_C + 3C\partial_D, \ldots.)\Phi = 0,$$

we shall find $V(-b\partial_M)\Phi = 0$ (every other term being annihilated by V). Thus

$$V(b\partial_M)\Phi = (\partial_M\Phi)\,Vb = 2a^2\partial_M\Phi = 0,$$

and consequently $\partial_M\Phi = 0$. Hence

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots.)\Phi = 0.$$

The equation $\partial_M\Phi = 0$ shows that M does not appear in the expression for any principiant in terms of the capital letters, while

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots.)\Phi = 0$$

shows that Φ is an invariant in $A, B, C, D, \ldots.$

We have thus shown that every invariant of

$$(A, B, C, \ldots.)(x, y)^j$$

is a principiant, and conversely that every principiant is an invariant of

$$(A, B, C, \ldots.)(x, y)^j,$$

or such an invariant multiplied by a power of a.

LECTURE XXIX.

From the theorem that every Principiant is (to a power of a près) an Invariant in the reciprocantive elements $A, B, C, \ldots.$ we readily deduce its correlative in which, everything else remaining unchanged, the *reciprocantive* elements $A, B, C, \ldots.$ are replaced by a set of *invariantive* elements which we call $A_0, A_1, A_2, \ldots.$ The equations connecting the new elements with the old ones are as follows:

$$A_0 = A,$$

$$A_1 = B - \left(\frac{b}{2}\right)A,$$

$$A_2 = C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A,$$

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

$$A_4 = E - 4\left(\frac{b}{2}\right)D + 6\left(\frac{b}{2}\right)^2 C - 4\left(\frac{b}{2}\right)^3 B + \left(\frac{b}{2}\right)^4 A,$$

. . . .

We have, in the first place, to prove that $A_0, A_1, A_2, \ldots$ are all of them invariants in the small letters $a, b, c, \ldots$ This is an immediate consequence of the identities

$$\Omega A = 0,$$
$$\Omega B = A \times \frac{a}{2},$$
$$\Omega C = 2B \times \frac{a}{2},$$
$$\cdots\cdots\cdots$$

established in the preceding Lecture, coupled with the fact that $\Omega b = a$. Thus

$$\Omega A_0 = \Omega A = 0,$$
$$\Omega A_1 = -\frac{b}{2}\,\Omega A + \left(\Omega B - A \times \frac{a}{2}\right) = 0,$$
$$\Omega A_2 = \left(\frac{b}{2}\right)^2 \Omega A - 2\left(\frac{b}{2}\right)\left(\Omega B - A \times \frac{a}{2}\right) + \left(\Omega C - 2B \times \frac{a}{2}\right) = 0;$$

and in general, writing the equation which gives A_n in the form

$$A_n = \left(-\frac{b}{2}\right)^n A + n\left(-\frac{b}{2}\right)^{n-1} B + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2} C + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3} D + \ldots,$$

and operating on it with Ω, we find

$$\Omega A_n = \left(-\frac{b}{2}\right)^n \Omega A + n\left(-\frac{b}{2}\right)^{n-1}\left(\Omega B - A \times \frac{a}{2}\right) + \frac{n(n-1)}{1.2}\left(-\frac{b}{2}\right)^{n-2}\left(\Omega C - 2B \times \frac{a}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}\left(-\frac{b}{2}\right)^{n-3}\left(\Omega D - 3C \times \frac{a}{2}\right) + \ldots$$
$$= 0 \text{ (each term vanishing separately).}$$

We next observe that

$(A_0, A_1, A_2, \ldots)(x, y)^j$, being equal to $(A, B, C, \ldots)\left(x - \frac{b}{2}y, y\right)^j$, is a linear transformation of $(A, B, C, \ldots)(x, y)^j$,

and that the determinant of the transformation $\begin{vmatrix} 1 & -\frac{b}{2} \\ 0 & 1 \end{vmatrix}$ is equal to unity.

Hence every invariant in $A_0, A_1, A_2, \ldots$ is equal to the corresponding invariant in $A, B, C, \ldots$, which proves the theorem in question.

Each of the invariantive elements $A_0, A_1, A_2, \ldots$ is, so to say, a *reciprocant* relative to the one which immediately precedes it, just as in the cognate

theorem each of the capital letters A, B, C, ... was an *invariant* relative to its antecedent. It is in fact easily seen that

$$\begin{aligned} VA_0 &= 0, \\ VA_1 &= -A_0a^2, \\ VA_2 &= -2A_1a^2, \\ VA_3 &= -3A_2a^2, \\ &\dots\dots\dots \end{aligned}$$

and in general $$VA_n = -nA_{n-1}a^2.$$

Thus, for example, if we operate with V on

$$A_3 = D - 3\left(\frac{b}{2}\right)C + 3\left(\frac{b}{2}\right)^2 B - \left(\frac{b}{2}\right)^3 A,$$

remembering that A, B, C, D are pure reciprocants, we shall find

$$VA_3 = -\frac{3}{2}\left\{C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A\right\} VB.$$

But $$C - 2\left(\frac{b}{2}\right)B + \left(\frac{b}{2}\right)^2 A = A_2 \text{ and } Vb = 2a^2;$$

so that $$VA_3 = -3A_2a^2.$$

In like manner, operating with V on

$$A_n = (A, B, C, \dots)\left(-\frac{b}{2}, 1\right)^n,$$

we obtain $$\begin{aligned} VA_n &= -\frac{n}{2}(A, B, C, \dots)\left(-\frac{b}{2}, 1\right)^{n-1} Vb \\ &= -nA_{n-1}a^2. \end{aligned}$$

This property enables us to give a proof (exactly similar to the proof of the cognate theorem in the preceding Lecture) of the theorem that every principiant is expressible as the product of an invariant in A_0, A_1, A_2, ... by a suitable power of a. We first observe that, using N to denote $ac - b^2$,

$$N, A_0, A_1, A_2, \dots$$

form a series of invariantive protomorphs of equal degree and weight.

Hence it follows that any invariant of degree i and weight w can be expressed in the form

$$a^{i-w}\Phi(N, A_0, A_1, A_2, \dots),$$

and consequently that every Principiant can be expressed in this form, provided only that $$V\Phi = 0.$$

Substituting for VA_0, VA_1, VA_2, ... their values given above, and at the same time observing that

$$VN = V(ac - b^2) = 5a^2b - 4a^2b = a^2b,$$

we find $$V\Phi = a^2(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots.)\Phi = 0.$$

Finally, we prove that Φ does not contain N, but is an invariant in A_0, A_1, A_2, ... alone, by operating with Ω on

$$(b\partial_N - A_0\partial_{A_1} - 2A_1\partial_{A_2} - 3A_2\partial_{A_3} - \ldots.)\Phi = 0,$$

when it is easily seen that every term vanishes except the first, which gives

$$\Omega(b\partial_N\Phi) = \Omega b \times \partial_N\Phi = 0,$$

where, $\Omega b = a$ being different from zero, we must have $\partial_N\Phi = 0$.

The invariants N, A_0, A_1, A_2, ... obey a law of successive derivation similar to that which holds for the reciprocants M, A, B, C, ...

Starting with $N = ac - b^2$ and operating continually with

$$G' = a\frac{d}{dx} - (3i + 2w)b = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c + \ldots.,$$

we shall find

$$\begin{aligned} G'N &= 5A_0, \\ G'A_0 &= 6A_1, \\ G'A_1 &= 7A_2 - NA_0, \\ G'A_2 &= 8A_3 - 2NA_1, \\ G'A_3 &= 9A_4 - 3NA_2, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

and generally $$G'A_n = (n+6)A_{n+1} - nNA_{n-1}.$$

These equations are exactly analogous to

$$\begin{aligned} GM &= 5A, \\ GA &= 6B, \\ GB &= 7C + MA, \\ GC &= 8D + 2MB, \\ GD &= 9E + 3MC, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

in which $M = ac - \frac{5}{4}b^2$, and GM, GA, GB, ... are the educts of M, A, B, ... obtained by operating with

$$G = a\frac{d}{dx} - (3i + w)b = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + \ldots.$$

It should be noticed that the two generators G and G' are connected by the relation $$G' = G - wb,$$

where w is the weight of the operand.

Also, that

$$Gb = 4(ac - b^2) = 4N, \text{ and } G'b = 4ac - 5b^2 = 4M.$$

We may easily verify that

$$G'N = 5A_0 = 5(a^2d - 3abc + 2b^3)$$

by operating with $G' = (4ac - 5b^2)\partial_b + (5ad - 7bc)\partial_c$ on $N = ac - b^2$.

To prove that $\qquad G'A_0 = 6A_1,$

we operate on $\qquad A_0 = A,$

for which the weight is 3, with

$$G' = G - 3b.$$

Thus $\qquad G'A_0 = (G - 3b)A = 6B - 3bA = 6A_1.$

For by definition $\qquad A_1 = B - \left(\frac{b}{2}\right)A.$

In general, to find $G'A_n$, we have by definition

$$A_n = (A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^n,$$

and, since the weight of A_n is $n + 3$,

$$G'A_n = GA_n - (n+3)bA_n.$$

Now,

$$GA_n = G(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$= (GA, GB, GC, \ldots .)\left(-\frac{b}{2}, 1\right)^n - \frac{n}{2}(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1} G$$

Substituting for GA, GB, GC, ... their known values, and remembering that $Gb = 4N$ and that $(A, B, C, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1} = A_{n-1}$, we have

$$GA_n = (6B, 7C, 8D, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots .)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}$$

$$= 6(B, C, D, \ldots .)\left(-\frac{b}{2}, 1\right)^n + (0, C, 2D, 3E, \ldots .)\left(-\frac{b}{2}, 1\right)^n$$

$$+ M(0, A, 2B, 3C, \ldots .)\left(-\frac{b}{2}, 1\right)^n - 2nNA_{n-1}.$$

But $\qquad (0, C, 2D, 3E, \ldots .)\left(-\frac{b}{2}, 1\right)^n$

$$= nC\left(-\frac{b}{2}\right)^{n-1} + n(n-1)D\left(-\frac{b}{2}\right)^{n-2}$$

$$+ \frac{n(n-1)(n-2)}{1.2}E\left(-\frac{b}{2}\right)^{n-3} + \ldots .$$

$$= n(C, D, E, \ldots .)\left(-\frac{b}{2}, 1\right)^{n-1};$$

and similarly

$$(0,\ A,\ 2B,\ 3C,\ \ldots)\left(-\frac{b}{2},\ 1\right)^n = n(A,\ B,\ C,\ \ldots)\left(-\frac{b}{2},\ 1\right)^{n-1} = nA_{n-1}.$$

Hence

$$GA_n = 6(B,\ C,\ D,\ \ldots)\left(-\frac{b}{2},\ 1\right)^n + n(C,\ D,\ E,\ \ldots)\left(-\frac{b}{2},\ 1\right)^{n-1} + n(M-2N)A_{n-1}.$$

Now let $\qquad U = (A,\ B,\ C,\ \ldots)(u,\ v)^n;$

then $\qquad \dfrac{dU}{du} = n(A,\ B,\ C,\ \ldots)(u,\ v)^{n-1},$

and $\qquad \dfrac{dU}{dv} = n(B,\ C,\ D,\ \ldots)(u,\ v)^{n-1};$

whence it follows that

$$(1) \qquad U = (A,\ B,\ C,\ \ldots)(u,\ v)^n = u(A,\ B,\ C,\ \ldots)(u,\ v)^{n-1} + v(B,\ C,\ D,\ \ldots)(u,\ v)^{n-1}.$$

Similarly, we see that

$$(2) \qquad (B,\ C,\ D,\ \ldots)(u,\ v)^n = u(B,\ C,\ D,\ \ldots)(u,\ v)^{n-1} + v(C,\ D,\ E,\ \ldots)(u,\ v)^{n-1}.$$

Writing $u = -\dfrac{b}{2}$ and $v = 1$ in the above equations, and remembering that

$$(A,\ B,\ C,\ \ldots)\left(-\frac{b}{2},\ 1\right)^n = A_n,$$

we obtain immediately from (1)

$$(B,\ C,\ D,\ \ldots)\left(-\frac{b}{2},\ 1\right)^{n-1} = A_n + \frac{b}{2}A_{n-1},$$

and then (2) gives

$$(C,\ D,\ E,\ \ldots)\left(-\frac{b}{2},\ 1\right)^{n-1} = \left(A_{n+1} + \frac{b}{2}A_n\right) + \frac{b}{2}\left(A_n + \frac{b}{2}A_{n-1}\right)$$
$$= A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}.$$

But it has been shown that

$$GA_n = 6(B,\ C,\ D,\ \ldots)\left(-\frac{b}{2},\ 1\right)^n + n(C,\ D,\ E,\ \ldots)\left(-\frac{b}{2},\ 1\right)^{n-1} + n(M-2N)A_{n-1}.$$

Hence, by substitution,

$$GA_n = 6\left(A_{n+1} + \frac{b}{2}A_n\right) + n\left(A_{n+1} + bA_n + \frac{b^2}{4}A_{n-1}\right) + n(M-2N)A_{n-1}$$
$$= (n+6)A_{n+1} + (n+3)bA_n + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1}.$$

Now,

$$G'A_n = GA_n - (n+3)bA_n$$
$$= (n+6)A_{n+1} + n\left(M + \frac{b^2}{4} - 2N\right)A_{n-1},$$

where $$M + \frac{b^2}{4} = ac - \frac{5}{4}b^2 + \frac{b^2}{4} = ac - b^2 = N.$$

Thus $$G'A_n = (n+6)\,A_{n+1} - nNA_{n-1},$$

which proves the law of successive derivation for the invariantive elements $A_0, A_1, A_2, \ldots$.*

We now proceed to explain the method of transforming a Principiant, given in terms of the small letters $a, b, c, \ldots$, into one expressed in terms of $a, A, B, C, \ldots$.

Remembering that the expressions for

$$A,\ B,\ C,\ D,\ E,\ \ldots$$

have for their most advanced small letters

$$d,\ e,\ f,\ g,\ h,\ \ldots,$$

and that, in each capital letter, the most advanced letter occurs only in the first degree, multiplied by a power of a, it follows, as an immediate consequence, that we may, by continually substituting for the most advanced letter, eliminate $d, e, f, g, h, \ldots$ from any rational integral function

$$\phi(a, b, c, d, e, f, g, h, \ldots)$$

and thus transform it into another function whose arguments are

$$a,\ b,\ c,\ A,\ B,\ C,\ D,\ E,\ \ldots$$

and which is rational in all its arguments, and integral in all of them, with the possible exception of the first argument, a.

But (see Lecture XXVIII) the result of this elimination is known to be

$$a^{i-w}\Phi(A, B, C, D, E, \ldots)$$

in the case where ϕ is a Principiant of known degree i and weight w. Hence b and c must disappear spontaneously during the process of elimination.

This being so, we can give b and c any arbitrary values, without thereby affecting the result, and it will greatly simplify the work to take $b = 0$ and $c = 0$.

It is also permissible to take $a = 1$; for, although the factor a^{i-w} is thereby lost, it can always be restored in the final result because both i and w are known

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* The establishment of the scale of relation between the terms of the $A_0, A_1, A_2, \ldots$ series and the above proof of it is due exclusively to Mr. Hammond. J. J. S.

numbers. Now, if we write $a=1$, $b=0$, $c=0$ in the known expressions for $A, B, C, D, \ldots$, we shall find

$$\begin{aligned} A &= d, \\ B &= e, \\ C &= f, \\ D &= g - \frac{25}{8} d^2, \\ E &= h - \frac{15}{2} de, \\ &\ldots\ldots\ldots \end{aligned}$$

Hence we have to eliminate $d, e, f, g, h, \ldots$ between the above equations and

$$P = \phi(1, 0, 0, d, e, f, g, h, \ldots),$$

where P stands for the given Principiant. In other words, we have to substitute for

$$\begin{array}{cccccccc} a, & b, & c, & d, & e, & f, & g \quad , & h \quad , \ldots \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8} A^2, & E + \frac{15}{2} AB, \ldots \end{array}$$

in

$$P = \phi(a, b, c, d, e, f, g, h, \ldots).$$

The result of this substitution will be

$$P = \Phi(A, B, C, D, E, \ldots),$$

where, to compensate for the factor lost by taking $a=1$, we must multiply Φ by a^{i-w}. As an easy example, consider the Principiant which Halphen calls Δ, and for which he obtains the expression

$$\begin{vmatrix} b & c & d & e & f \\ a & b & c & d & e \\ -a^2 & 0 & b^2 & 2bc & 2bd + c^2 \\ 0 & a^2 & 2ab & 2ac + b^2 & 2ad + 2bc \\ 0 & 0 & a^2 & 3ab & 3b^2 + 3ac \end{vmatrix}.$$

Here the degree $i = 8$ and the weight $w = 8$; so that $i - w = 0$, and no factor has to be restored. On making the substitutions spoken of, the determinant becomes

$$\begin{vmatrix} 0 & 0 & A & B & C \\ 1 & 0 & 0 & A & B \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2A \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

which immediately reduces to $AC - B^2$ by striking out the first three columns and the last three rows.

Of this Principiant we shall have more to say hereafter.

LECTURE XXX.

The method of substituting large letters for small ones will be better understood if we employ it to obtain an expression of the form

$$a^{i-w}\Phi(M, A, B, C, D, E, \ldots .)$$

for any pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots .)$$

of known degree i and weight w in the small letters.

The transformation is effected by substituting in ϕ for $c, d, e, f, g, h, \ldots .$ their values (which are perfectly definite) in terms of $a, b, M, A, B, C, D, E, \ldots .$ But since b does not appear in the final result, we are at liberty to give it any arbitrary value, and it will be convenient to take $b=0$, for then (see Lecture XXVIII) we have

$$\begin{aligned}
M &= ac, \\
A &= a^2d, \\
B &= a^3e - 2a^2c^2, \\
C &= a^4f - 5a^3cd, \\
D &= a^5g - \frac{25}{8}a^4d^2 - 6a^4ce + 7a^3c^3, \\
E &= a^6h - \frac{15}{2}a^5de - 7a^5cf + 29a^4c^2d, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

There is an additional advantage in taking $b=0$, viz. that then the values of the *invariants* $N, A_0, A_1, A_2, \ldots .$ (see their definition at the beginning of Lecture XXIX) exactly coincide with those of the *reciprocants* $M, A, B, C, \ldots .$ set forth above. Hence, merely interchanging the capital letters, the same substitutions enable us to express any invariant in terms of $a, N, A_0, A_1, \ldots .$, as well as any reciprocant in terms of $a, M, A, B, \ldots .$

The solution of the above equations will give $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots .$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots .$; but we can, without loss of generality, put $a=1$, when we shall find

$$\begin{aligned}
a &= 1, \\
b &= 0, \\
c &= M, \\
d &= A, \\
e &= B + 2M^2, \\
f &= C + 5MA, \\
g &= D + \frac{25}{8}A^2 + 6MB + 5M^3, \\
h &= E + \frac{15}{2}AB + 7MC + 6MA^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The substitution of these values in the pure reciprocant

$$\phi(a, b, c, d, e, f, g, h, \ldots.)$$

will convert it into

$$\Phi(M, A, B, C, D, E, \ldots.).$$

We have written $a=1$ for the sake of simplicity; but without doing this we have, since ϕ is homogeneous of degree i,

$$\phi(a, 0, c, d, e, \ldots.) = a^i\phi\left(1, 0, \frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots.\right).$$

Hence, substituting for $\frac{c}{a}, \frac{d}{a}, \frac{e}{a}, \ldots.$ in terms of $\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots.,$

$$\phi(a, 0, c, d, e, \ldots.) = a^i\Phi\left(\frac{M}{a^2}, \frac{A}{a^3}, \frac{B}{a^4}, \ldots.\right);$$

or, since $M, A, B, \ldots.$ are of weights $2, 3, 4, \ldots.,$ and Φ is of weight w,

$$\phi(a, 0, c, d, e, \ldots.) = a^{i-w}\Phi(M, A, B, \ldots.).$$

Thus, in consequence of writing $a=1$, the factor a^{i-w} has been lost; but this factor can always be restored, both i and w being known numbers.

When ϕ is a Principiant, M will not appear in the final result, which will be identical with that obtained by the simpler substitutions of the preceding Lecture. If, for example, we substitute for

$$\begin{array}{llllll} a, & b, & c, & d, & e\quad, & f\quad, \\ 1, & 0, & M, & A, & B+2M^2, & C+5MA, \end{array}$$

instead of

$$\begin{array}{llllll} 1, & 0, & 0, & A, & B\quad, & C\quad, \end{array}$$

in the determinant expression for Halphen's Δ, previously given, it becomes

$$\begin{vmatrix} 0 & M & A & B+2M^2 & C+5MA \\ 1 & 0 & M & A & B+2M^2 \\ -1 & 0 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 2M & 2A \\ 0 & 0 & 1 & 0 & 3M \end{vmatrix}.$$

Subtracting the 4th row multiplied by M from the first, the determinant reduces to

$$\begin{vmatrix} 0 & A & B & C+3MA \\ 1 & M & A & B+2M^2 \\ -1 & 0 & 0 & M^2 \\ 0 & 1 & 0 & 3M \end{vmatrix}.$$

Again, subtracting the 2[d] column multiplied by $3M$ from the last, and reducing, the determinant becomes

$$\begin{vmatrix} 0, & B, & C \\ 1, & A, & B - M^2 \\ -1, & 0, & M^2 \end{vmatrix} = AC - B^2,$$

where M disappears, as it ought to do, because Δ is a Principiant.

In what follows we shall have frequent occasion to make use of the fact that if R_a is an absolute pure reciprocant, $\dfrac{dR_a}{a^{\frac{1}{3}}dx}$, which we know is a pure reciprocant, is also an absolute one.

This is very easily proved. For let R be any pure reciprocant, of degree i and weight w, which becomes R_a when made absolute by division by a power of a, then

$$R_a = \frac{R}{a^{\frac{\mu}{3}}}, \text{ where } \mu = 3i + w,$$

and, using G as usual to denote the generator for pure reciprocants,

$$\frac{dR_a}{dx} = \frac{GR}{a^{\frac{\mu}{3}+1}}.$$

Hence

$$\frac{dR_a}{a^{\frac{1}{3}}dx} = \frac{GR}{a^{\frac{\mu+4}{3}}},$$

which is an absolute pure reciprocant because GR, which is of degree $i + 1$ and weight $w + 1$, must be divided by $a^{\frac{\mu+4}{3}}$ in order to make it absolute. Thus, if M_a, A_a, B_a, C_a, ... are what M, A, B, C, ... become when each of them is made absolute by division by a power of a, we have

$$a^{-\frac{1}{3}} \frac{d}{dx} M_a = 5A_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} A_a = 6B_a,$$

$$a^{-\frac{1}{3}} \frac{d}{dx} B_a = 7C_a + M_a A_a,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

We shall use these results in deducing the complete primitive of the differential equation

$$AC - B^2 = 0$$

from that of the equation in pure reciprocants,

$$25A^2 - 16M^3 = 0.$$

This equation may be written in the form

$$25A_a^2 = 16M_a^3;$$

whence, by differentiation, we obtain

$$50A_a\left(a^{-\frac{1}{3}}\frac{d}{dx}A_a\right) = 48M_a^2\left(a^{-\frac{1}{3}}\frac{d}{dx}M_a\right),$$

which gives
$$50A_a \cdot 6B_a = 48M_a^2 . 5A_a;$$

i. e.
$$5B_a = 4M_a^2.$$

Differentiating this result, we find

$$5(7C_a + M_aA_a) = 40M_aA_a;$$

which gives
$$C_a = M_aA_a.$$

We now restore the non-absolute reciprocants M, A, B, C; *i. e.* we write

$$5B = 4M^2 \text{ and } C = MA.$$

Hence $25(AC - B^2) = M(25A^2 - 16M^3) = 0$ (because $25A^2 = 16M^3$).

Now, the equation $AC - B^2 = 0$ remains unaltered by any homographic substitution, so that it will be satisfied not only by any solution of the equation in pure reciprocants $25A^2 - 16M^3 = 0$, but also by any homographic transformation of such solution. But it has been shown (in Lecture XIII, *American Journal of Mathematics*, Vol. IX, p. 16) that the complete primitive of $25A^2 - 16M^3 = 0$ is a linear transformation of $y = x^\lambda$, where $\lambda^2 - \lambda + 1 = 0$ (*i. e.* where λ is a cube root of negative unity).

Consequently any homographic transformation of $y = x^\lambda$ is a solution of

$$AC - B^2 = 0.$$

Moreover, this is its complete primitive; for the highest letter, f, which occurs in $AC - B^2$, corresponds to the seventh order of differentiation, and if we write

$$y = \frac{Y}{Z}, \quad x = \frac{X}{Z},$$

where X, Y, Z are general linear functions of x, y, 1 (*i. e.* if we make the most general homographic substitution), $y = x^\lambda$ becomes $Y = X^\lambda Z^{1-\lambda}$, which will be found to contain exactly 7 independent arbitrary constants. Thus the complete primitive of $AC - B^2 = 0$ is $Y = X^\lambda Z^{1-\lambda}$, where X, Y, Z are general linear functions of x, y, 1, and λ is a cube root of negative unity.

Observe that although any solution of $M = 0$ also makes A, B, C, ... all vanish, and so satisfies $AC - B^2 = 0$, we cannot from this infer that a homographic transformation of the parabola $y = x^2$ will be the complete primitive of

$AC - B^2 = 0$. For, though $YZ = X^2$ is a solution of $AC - B^2 = 0$, it only contains 5 independent arbitrary constants, and therefore cannot be its complete primitive. Neither can $YZ = X^2$ be obtained from the complete primitive by giving special values to the arbitrary constants. Hence $YZ = X^2$ is a singular solution of $AC - B^2 = 0$.

We may also deduce the differential equation of the curve $Y = X^\lambda Z^{1-\lambda}$, where λ has a general value, from the corresponding equation in pure reciprocants,

$$25(2\lambda^2 - 5\lambda + 2)A^2 + 16(\lambda + 1)^2 M^3 = 0,$$

which has (see *American Journal of Mathematics*, Vol. IX, p. 14) for its complete primitive any linear transformation of the general parabola $y = x^\lambda$.

Writing for shortness

$$2\lambda^2 - 5\lambda + 2 = p \text{ and } (\lambda + 1)^2 = q,$$

and at the same time making both A and M absolute, the above equation becomes

$$25pA_a^2 + 16qM_a^3 = 0.$$

Hence, by differentiation, we obtain

$$50pA_a . 6B_a + 48qM_a^2 . 5A_a = 0,$$

which gives

$$5pB_a + 4qM_a^2 = 0.$$

After a second differentiation we find

$$5p(7C_a + M_aA_a) + 40qM_aA_a = 0;$$

i. e.

$$7pC_a + (p + 8q)M_aA_a = 0.$$

We now replace the absolute reciprocants M_a, A_a, B_a, C_a by M, A, B, C, and thus write the original equation and its two differentials in the form

$$25pA^2 = -16qM^3,$$
$$5pB = -4qM^2,$$
$$7pC = -(p + 8q)MA.$$

Hence we find

$$5^2.7.p^2(AC - B^2) = -25p(p + 8q)MA^2 - 16.7q^2M^4$$
$$= 16q(p + q)M^4,$$
$$5^6.7^3.p^6(AC - B^2)^3 = 16^3q^3(p + q)^3M^{12},$$
$$5^8p^4A^8 = 16^4q^4M^{12},$$

and, eliminating M from the two last equations,

$$2^4.7^3.p^2q(AC - B^2)^3 = 5^2(p + q)^3A^8.$$

Now restoring $\quad p = 2\lambda^2 - 5\lambda + 2 = (\lambda - 2)(2\lambda - 1)$

and $\quad q = (\lambda + 1)^2,$

we have $\quad p + q = 3(\lambda^2 - \lambda + 1);$

so that the final equation becomes

$$2^4.7^3(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2(AC-B^2)^3=3^3.5^2(\lambda^2-\lambda+1)^3A^8.$$

The same reasoning as before will show that, for a general value of λ, the complete primitive of this equation is the general homographic transformation $Y=X^\lambda Z^{1-\lambda}$ of the curve $y=x^\lambda$.

There is, however, a special exceptional case in which the differential equation becomes $$2^6.7^3(AC-B^2)^3=3^3.5^2A^8,$$
the corresponding value of the parameter λ being either 0, 1 or ∞, as may be seen by solving the equation

$$(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2=4(\lambda^2-\lambda+1)^3.$$

In the case where $\lambda=0$ or ∞ we can, in the same manner as before, show that the complete primitive is a homographic transformation of the curve $y=e^x$ by deducing the differential equation from the corresponding equation in pure reciprocants, $$25A^2+8M^3=0,$$
whose complete primitive is (see Lecture XIII) a linear transformation of $y=e^x$.

When $\lambda=1$ the corresponding equation in pure reciprocants is

$$25A^2-64M^3=0,$$

whose complete primitive may be shown to be a linear transformation of $y=x\log x$. The reason why these two distinct equations in pure reciprocants lead to the same equation in principiants is that the two curves $y=e^x$ and $y=x\log x$ are *homographically* equivalent but not *linearly* transformable into one another. For we may write the equation $y=x\log x$ in the form $x=e^{\frac{y}{x}}$, which is a homographic transformation of $y=e^x$.

Besides the special case just considered, in which the complete primitive of the equation in Principiants is $\frac{Y}{Z}=e^{\frac{X}{Z}}$, we may notice that in which the parameter λ is either -1, 2, or $\frac{1}{2}$, the differential equation reducing to $A=0$ simply, and its complete primitive $Y=X^\lambda Z^{1-\lambda}$ being the equation to a conic, as it should be. The case where $\lambda^2-\lambda+1=0$ and the differential equation reduces to $AC-B^2=0$ has been considered already. There remains the case in which $\lambda=3$, when the complete primitive becomes $YZ^2=X^3$ (the equation of the general cuspidal cubic) and the differential equation assumes the simple form $$\left(\frac{AC-B^2}{3}\right)^3=\left(\frac{A}{2}\right)^8,$$
which is therefore the differential equation of cuspidal cubics.

We shall hereafter show that in this case the Principiant

$$2^8(AC-B^2)^3-3^3A^8,$$

which is apparently of the 24^{th} degree, loses a factor a^4 and so sinks to the 20^{th} degree. It is, however, generally difficult to determine the power of a contained as a factor in a Principiant given in terms of the large letters.

The results obtained in the present Lecture agree with those of M. Halphen contained in his Thèse sur les Invariants différentiels (Paris, Gauthier-Villars, 1878), which contains a complete investigation of the properties of the Principiant $AC-B^2$, which he calls Δ. But our point of view is different from his. He obtains Δ in the form of a determinant from geometrical considerations. With him $\Delta=0$ is the differential equation which expresses the condition that, at a point x, y on any curve, a nodal cubic shall exist, having its node at x, y, and such that *one of its branches* shall have 8-point contact with the curve at that point. With us $AC-B^2$ is the simplest example, after the Mongian A, of an invariant in the capital letters A, B, C, ...

LECTURE XXXI.

We may include λ among the arbitrary constants in the primitive equation $Y=X^\lambda Z^{1-\lambda}$, which can also be written in the form

$$\lambda\log X-\log Y+(1-\lambda)\log Z=0,$$

or (X, Y, Z being general linear function of x, y, 1) in the equivalent form

$$\lambda\log(y+\alpha x+\beta)-\log(y+\alpha' x+\beta')+(1-\lambda)\log(y+\alpha'' x+\beta'')=\text{const.},$$

which evidently contains 8 independent arbitrary constants.

One of these will be made to disappear by differentiation, and thus we shall obtain a differential equation of the first order, containing 7 arbitrary constants, identical (when the constants are rearranged) with

$$(y-xt)(lx+my)+t(l'x+m'y+n')+l''x+m''y+n''=0,$$

which is known as Jacobi's Equation.

For, by differentiating the primitive equation, we obtain

$$\lambda(t+\alpha)(y+\alpha x+\beta)^{-1}-(t+\alpha')(y+\alpha' x+\beta')^{-1}$$
$$+(1-\lambda)(t+\alpha'')(y+\alpha'' x+\beta'')^{-1}=0,$$

which, when cleared of negative indices by multiplication, becomes

$$\lambda(y+\alpha'x+\beta')\{(y+\alpha''x+\beta'')(t+\alpha)-(y+\alpha x+\beta)(t+\alpha'')\}$$
$$+(y+\alpha x+\beta)\{(y+\alpha'x+\beta')(t+\alpha'')-(y+\alpha''x+\beta'')(t+\alpha')\}=0.$$

Writing this equation in the equivalent form

$$\lambda(y+\alpha'x+\beta')\{(\alpha-\alpha'')(y-xt)+(\beta''-\beta)t+(\alpha\beta''-\alpha''\beta)\}$$
$$+(y+\alpha x+\beta)\{(\alpha''-\alpha')(y-xt)+(\beta'-\beta'')t+(\alpha''\beta'-\alpha'\beta'')\}=0,$$

it is easily seen to be identical with Jacobi's equation given above.

The seven arbitrary constants which occur in Jacobi's equation are the mutual ratios of the eight coefficients l, m, l', m', n', l'', m'', n'', any one of which may have an arbitrarily chosen value assigned to it.

Taking $m=-1$, the equation may be written in the form

$$Pt+lxy-y^2+l''x+m''y+n''=0,$$

where $$P=l'x+m'y+n'-lx^2+xy.$$

In order to eliminate n'' and l'', we differentiate the above equation twice. The first differentiation gives

$$2aP+t(P'+lx-2y+m'')+ly+l''=0,$$

where $P'=\dfrac{dP}{dx}=l'+m't-2lx+y+xt$, and the second differentiation gives

$$6bP+2a(2P'+lx-2y+m'')+t(P''+2l-2t)=0.$$

Now, $P''=\dfrac{dP'}{dx}=2a(m'+x)+2(t-l)$; so that, on substituting this value, the above equation becomes

$$3bP+aQ=0, \tag{1}$$

where $$Q=2P'+lx-2y+m''+m't+xt$$
$$=2l'+3m't-3lx+3xt+m''.$$

Differentiating (1) we have

$$12cP+3bP'+3bQ+aQ'=0,$$

where $Q'=3(t-l)+6a(x+m')=3R+6aS$, suppose.

Thus we have $$4cP+bP'+bQ+aR+2a^2S=0. \tag{2}$$

Differentiating this 4 times in succession, and at each step substituting for

$$P'',\quad Q',\quad R',\quad S',$$

their values $$2R+2aS,\quad 3R+6aS,\quad 2a,\quad 1,$$

we obtain 4 more equations, from which, combined with the 2 previously obtained, we can eliminate

$$P,\ P',\ Q,\ R,\ S.$$

Thus, differentiating (2), we find

$$20dP + 8cP' + b(2R + 2aS) + 4cQ + b(3R + 6aS) + 3bR + 2a^2 + 12abS + 2a^2 = 0;$$

i. e.

$$5dP + 2cP' + cQ + 2bR + 5abS + a^2 = 0, \tag{3}$$

and continuing the same process,

$$6eP + 3dP' + dQ + 3cR + (6ac + 3b^2)S + 3ab = 0, \tag{4}$$

$$7fP + 4eP' + eQ + 4dR + (7ad + 7bc)S + (4ac + 2b^2) = 0, \tag{5}$$

$$8gP + 5fP' + fQ + 5eR + (8ae + 8bd + 4c^2)S + (5ad + 5bc) = 0. \tag{6}$$

The result of elimination is

$$\begin{vmatrix} 3b & 0 & a & 0 & 0 & 0 \\ 4c & b & b & a & 2a^2 & 0 \\ 5d & 2c & c & 2b & 5ab & a^2 \\ 6e & 3d & d & 3c & 6ac + 3b^2 & 3ab \\ 7f & 4e & e & 4d & 7ad + 7bc & 4ac + 2b^2 \\ 8g & 5f & f & 5e & 8ae + 8bd + 4c^2 & 5ad + 5bc \end{vmatrix} = 0,$$

where the determinant equated to zero is a Principiant.

In his Thèse sur les Invariants différentiels, p. 42, M. Halphen states that this equation can be found by eliminating the constants from Jacobi's equation, but he does not set out the work. When in the above determinant twice the 3d column is added to the second, it becomes exactly identical with the one given by Halphen, which he calls T.

We proceed to express the above result in terms of the capital letters, using the method explained in Lecture XXIX, and observing that the determinant is of degree 8 and of weight 12; so that in this case $i - w = 8 - 12 = -4$, showing that the final result has to be multiplied by a^{-4}.

Substituting in the determinant for

$$\begin{matrix} a & b & c & d & e & f & g \\ 1 & 0 & 0 & A & B & C & D + \frac{25}{8}A^2, \end{matrix}$$

it becomes

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 5A & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & 7A & 0 \\ 8D + 25A^2 & 5C & C & 5B & 8B & 5A \end{vmatrix}.$$

Subtracting the last column multiplied by $5A$ from the first, and the 4th column multiplied by 2 from the 5th, and then striking out rows and columns, we obtain

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & A & 0 & 0 & 0 \\ 7C & 4B & B & 4A & -A & 0 \\ 8D & 5C & C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 & 0 \\ 7C & 4B & 4A & -A & 0 \\ 8D & 5C & 5B & -2B & 5A \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 6B & 3A & 0 & 0 \\ 7C & 4B & -A & 0 \\ 8D & 5C & -2B & 5A \end{vmatrix} = \begin{vmatrix} 6B & 3A & 0 \\ 7C & 4B & A \\ 8D & 5C & 2B \end{vmatrix}$$

$$= 24\,(A^2D - 3ABC + 2B^3).$$

If, using Halphen's notation, we call the principiant now under consideration T, what we have proved is that

$$T = 24a^{-4}(A^2D - 3ABC + 2B^3),$$

and consequently that $A^2D - 3ABC + 2B^3$ is divisible by a^4.

The differential equation $T = 0$ corresponds, as we have seen, to the complete primitive $Y = X^\lambda Z^{1-\lambda}$, in which λ is counted as one of the arbitrary constants.

This result may be otherwise obtained. For we have shown in the preceding Lecture that the differential equation of the seventh order, from which all the arbitrary constants except λ have disappeared, has the form

$$(AC - B^2)^3 = \varkappa A^8,$$

where $\varkappa$ depends solely on λ.

Writing this equation in the form

$$(AC - B^2)\,A^{-\frac{8}{3}} = \text{const.},$$

and differentiating with respect to x, we remove the remaining arbitrary con-

stant, and thus obtain the differential equation of the 8^{th} order free from all arbitrary constants, a result which, to a factor près, must coincide with

$$T = 0.$$

We proceed to show how this differentiation may be performed without introducing any of the small letters. In the first place, it is clear that since

$$G = 4(ac - b^2)\partial_b + 5(ad - bc)\partial_c + 6(ae - bd)\partial_d + \ldots .$$

does not contain ∂_a and is linear in the other differential reciprocals $\partial_b, \partial_c, \ldots,$

$$\begin{aligned} Ga^\theta\Phi(A, B, C, \ldots) &= a^\theta G\Phi(A, B, C, \ldots) \\ &= a^\theta\left(\frac{d\Phi}{dA}GA + \frac{d\Phi}{dB}GB + \frac{d\Phi}{dC}GC + \ldots\right). \end{aligned}$$

And since we have

$$\begin{aligned} GA &= 6B, \\ GB &= 7C + MA, \\ GC &= 8D + 2MB, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

it follows immediately that

$$\begin{aligned} Ga^\theta\Phi(A, B, C, \ldots) = a^\theta(6B\partial_A + 7C\partial_B + 8D\partial_C + \ldots)\Phi \\ + a^\theta M(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi. \end{aligned}$$

This is true for any function of the capital letters, whatever its nature may be; but when Φ is a principiant, it is also an invariant in the large letters; so that in this case we have

$$(A\partial_B + 2B\partial_C + 3C\partial_D + \ldots)\Phi = 0$$

and

$$Ga^\theta\Phi = a^\theta(6B\partial_A + 7C\partial_B + 8D\partial_C + \ldots)\Phi.$$

Now, the operation of G on a function of degree i and weight w is equivalent to that of $a\dfrac{d}{dx} - (3i + w)b$, or to that of $a\dfrac{d}{dx}$, when both $i = 0$ and $w = 0$ (which happens in the case of a plenarily absolute form). Hence, if we suppose Φ to be a plenarily absolute principiant, $G\Phi$ is also a principiant, though not a plenarily absolute one.

For a is a principiant, and $\dfrac{d\Phi}{dx}$ is a principiant; therefore $a\dfrac{d\Phi}{dx}$ or $G\Phi$ is one also.* Thus

$$6B\partial_A + 7C\partial_B + 8D\partial_C + \ldots,$$

*See the concluding paragraph of Lecture XXV, where it was shown that P, being a principiant (of degree i and weight w), $a\dfrac{dP}{dx} - (3i + w)bP$ is a reciprocant, and $a\dfrac{dP}{dx} - (3i + 2w)bP$ an invariant. This proves, what we omitted to mention there, that P being a *zero-weight* principiant,

$$GP = \left(a\frac{d}{dx} - 3ib\right)P \text{ is a principiant.}$$

It may here be remarked that a principiant of degree i and of *zero weight* is equal to the corresponding plenarily absolute principiant (which is a function of the large letters only) multiplied by the factor a^i, on which the operator G does not act.

acting on any plenarily absolute principiant, generates another principiant, but not a plenarily absolute one.

We now resume the consideration of the equation

$$(AC - B^2)\, A^{-\frac{8}{3}} = \text{const.}$$

Differentiating and multiplying by a, we have

$$a\frac{d}{dx}\left\{(AC - B^2)\, A^{-\frac{8}{3}}\right\} = 0.$$

Hence, by what precedes,

$$(6B\partial_A + 7C\partial_B + 8D\partial_C)\{(AC - B^2)\, A^{-\frac{8}{3}}\} = 0;$$

or, using Θ to denote the operator,

$$6B\partial_A + 7C\partial_B + 8D\partial_C + \ldots\ldots,$$

$$A^{-\frac{8}{3}}\Theta(AC - B^2) - \frac{8}{3}A^{-\frac{11}{3}}(AC - B^2)\Theta A = 0;$$

or, observing that $\Theta A = 6B$,

$$A\Theta(AC - B^2) - 16B(AC - B^2) = 0.$$

This gives $\qquad A(6BC - 14BC + 8AD) - 16B(AC - B^2) = 0;$

or finally $\qquad A^2D - 3ABC + 2B^3 = 0.$

We may find a generator for principiants expressed in terms of the large letters similar to the expression for the reciprocant generator G in terms of the small letters. For let P be any principiant, of weight w, which, when reduced to zero weight by division by $A^{\frac{w}{3}}$, becomes $PA^{-\frac{w}{3}}$; then

$$\Theta\left(PA^{-\frac{w}{3}}\right)$$

is a principiant. But

$$\Theta\left(PA^{-\frac{w}{3}}\right) = A^{-\frac{w}{3}-1}(A\Theta - 2wB)\,P,$$

where, remembering that $A^{-\frac{w}{3}-1}$ is a principiant, $(A\Theta - 2wB)\,P$ is one also.

Now, the weights of $\qquad A,\ B,\ C,\ D, \ldots.$

being $\qquad 3,\ 4,\ 5,\ 6, \ldots.,$

we may write

$$w = 3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots\ldots,$$

and consequently

$$\begin{aligned} A\Theta - 2wB &= A(6B\partial_A + 7C\partial_B + 8D\partial_C + 9E\partial_D + \ldots.) \\ &\quad - 2B(3A\partial_A + 4B\partial_B + 5C\partial_C + 6D\partial_D + \ldots.) \\ &= (7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C + (9AE - 12BD)\partial_D + \ldots\ldots, \end{aligned}$$

which is the generator in question.

As an easy example of its use, suppose it to operate on $AC - B^2$; then

$$\begin{aligned}&\{(7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C\}(AC - B^2)\\ &= -2B(7AC - 8B^2) + A(8AD - 10BC)\\ &= 8(A^2D - 3ABC + 2B^3).\end{aligned}$$

The generator just obtained,

$$(7AC - 8B^2)\partial_B + (8AD - 10BC)\partial_C + (9AE - 12BD)\partial_D + \ldots.,$$

is a linear combination of Cayley's two generators (given in Lecture IV, Vol. VIII, p. 222 of this Journal), which, when we write A, B, C, ... instead of the corresponding small letters, become

$$(AC - B^2)\partial_B + (AD - BC)\partial_C + (AE - BD)\partial_D + \ldots.$$

and

$$(AC - 2B^2)\partial_B + (2AD - 4BC)\partial_C + (3AE - 6BD)\partial_D + \ldots.$$

Thus we shall obtain the principiant generator by adding the second of Cayley's generators to six times the first. Either of Cayley's generators acting on a principiant would of course give an invariant in the large letters (*i. e.* a principiant), but the combination we have used has special relation to the theory of the generation of principiants by differentiation.

LECTURE XXXII.

I will now pass on to the consideration of the Principiant which, when equated to zero, gives the Differential Equation to the most general Algebraic Curve of any order.

The Differential Equation to a Conic (see the reference given on p. 18, Vol. IX of this Journal) was obtained by Monge in the first decade of this century. This was followed by the determination, in 1868, by Mr. Samuel Roberts, of the Differential Equation to the general Cubic (see Vol. X, p. 47 of Mathematical Questions and Solutions from the Educational Times). I do not consider that any substantial advance was made upon this by Mr. Muir, in the Philosophical Magazine for February, 1886, except that he sets out explicitly the quantities to be eliminated in obtaining the final result. These may of course be collected from the processes indicated by Mr. Roberts, but are not set forth by him. In speaking of the history of this part of the subject, I pass over M. Halphen's

process for obtaining the Differential Equation to a Conic. It is very ingenious, like everything that proceeds from his pen, but, being founded on the solution of a quadratic equation, does not admit of being extended to forms of a higher degree, and consequently, viewed in the light of subsequent experience, must be regarded as faulty in point of method.

Let the Differential Equation to a curve of any order, when written in its simplest form, containing no extraneous factor, be $\chi = 0$. It is convenient to give χ a single name; I call it the Criterion. The integral of the Criterion to a curve of order n must contain as many arbitrary constants as there are ratios between the coefficients of a curve of the n^{th} order. The number of these ratios being $\frac{n^2+3n+2}{2} - 1$, the order of the Criterion ought to be $\frac{n^2+3n}{2}$.

It must be independent of Perspective Projection, because projection does not affect the order of a curve. Hence it is a Principiant, and as such ought not (when y is regarded as the dependent and x as the independent variable) to contain either x, y or $\frac{dy}{dx}$ (see Lecture XXIV, *American Journal of Mathematics*, Vol. IX, p. 155).

Let $U = 0$ be an algebraical equation of the n^{th} order between x, y. I write symbolically
$$U = (p + qx + y)^n = u^n,$$
where the different powers and products of p, q, 1 which occur in the expansion of u^n are considered as representing the different coefficients in U; so that, *ex. gr.*, if $n = 3$ the coefficients of
$$y^3,\ 3y^2x,\ 3y^2,\ 3yx^2,\ 6yx,\ 3y,\ x^3,\ 3x^2,\ 3x,\ 1$$
are represented by
$$1,\ q,\ p,\ q^2,\ pq,\ p^2,\ q^3,\ pq^2,\ p^2q,\ p^3.$$
The number of terms in U is
$$1 + 2 + 3 + \ldots . + (n+1) = \frac{(n+1)(n+2)}{2}.$$
The number of these containing y is
$$1 + 2 + 3 + \ldots . + n = \frac{n(n+1)}{2}.$$

To obtain the Differential Equation we equate to zero the Differential Derivatives of U of all orders from $n+1$ to $\frac{n^2+3n}{2}$ inclusive, and from the $\frac{n^2+n}{2}$ equations thus formed eliminate the $\frac{n^2+n}{2}$ coefficients of the terms in U containing y.

All the coefficients of pure powers of x will obviously disappear under differentiation; for no power of x higher than x^n occurs in U, and no differential derivative of U of lower order than $n+1$ is taken.

We thus find a differential equation of the order $\frac{n^2+3n}{2}$, free from all the $\frac{n^2+3n+2}{2}$ coefficients of U. This equation might conceivably contain x, y and *all* the successive differential derivatives of y with respect to x. But we know *à priori* that it ought not to contain either x, y or $\frac{dy}{dx}$; and in fact we shall be able so to conduct the elimination that x, y and $\frac{dy}{dx}$ appear only in the quantities to be eliminated and not in the final result.

Treating $u = p + qx + y$ as an ordinary algebraical quantity, we have, by Taylor's theorem,

$$\frac{1}{1.2.3\ldots.r}\cdot\frac{d^r u^n}{dx^r} = \text{co. } h^r \text{ in } \left(u + u_1 h + u_2\frac{h^2}{1.2} + u_3\frac{h^3}{1.2.3} + \ldots.\right)^n,$$

where $u_1, u_2, u_3, \ldots.$ are the successive differential derivatives of u with respect to x. And this result will remain true when for u^n we write U, meaning thereby that $\frac{1}{1.2.3\ldots.r}\cdot\frac{d^r U}{dx^r}$ will be the quantitative interpretation of the function of $u, u_1, u_2, \ldots.$ which multiplies h^r in the expansion of

$$\left(u + u_1 h + u_2\frac{h^2}{1.2} + \ldots.\right)^n,$$

subject to the condition that this function shall be *linear* in the coefficients of U. This condition can be fulfilled in only one way, so that there is no ambiguity in such interpretation. Hence the equations obtained by equating to zero the successive differential derivatives of U of all orders from $n+1$ to $\frac{n^2+3n}{2}$ inclusive may be written under the form

$$\text{co. } h^r \text{ in } \left(u + u_1 h + u_2\frac{h^2}{1.2} + u_3\frac{h^3}{1.2.3} + \ldots.\right)^n = 0,$$

where $$r = n+1,\ n+2,\ n+3, \ldots. \frac{n^2+3n}{2}.$$

Now, using $y_1, y_2, y_3, \ldots.$ to denote the successive differential derivatives of y with respect to x, we have

$$u_1 = q + y_1,\ u_2 = y_2,\ u_3 = y_3, \ldots.,$$

and, in general, $u_i = y_i$ when i is any positive integer greater than 1. Thus

$$\text{co. } h^r \text{ in } \left(u + u_1 h + y_2 \frac{h^2}{1.2} + y_3 \frac{h^3}{1.2.3} + \ldots .\right)^n = 0;$$

or, employing the usual modified derivatives $a, b, c, \ldots .,$

$$\text{co. } h^r \text{ in } (u + u_1 h + ah^2 + bh^3 + ch^4 + \ldots .)^n = 0.$$

Writing now $$Q = ah^2 + bh^3 + ch^4 + \ldots .,$$

and expanding $(u + u_1 h + Q)^n$ in ascending powers of Q, we have

$$\text{co. } h^r \text{ in } \left\{(u + u_1 h)^n + n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots .\right\} = 0,$$

where, remembering that $r > n$, the value of co. h^r in $(u + u_1 h)^n$ is zero; so that, omitting this term, we may write

$$\text{co. } h^r \text{ in } \left\{n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots . + Q^n\right\} = 0.$$

The quantities to be eliminated will now be combinations of the various powers of u, u_1 and 1. Their number will be the same as that of the terms in $(u, u_1, 1)^{n-1}$, which is $\frac{n^2 + n}{2}$, the same number as that of the equations between which the elimination is to be performed.

We now use $(m.\mu)$ to denote the coefficient of h^m in Q^μ (which, since

$$Q = ah^2 + bh^3 + ch^4 + \ldots .,$$

will be independent of the combinations of u and u_1 to be eliminated), and in writing out the $\frac{n^2 + n}{2}$ equations which result from making the coefficients of $h^{n+1}, h^{n+2}, \ldots . h^{\frac{n^2+3n}{2}}$ in

$$n(u + u_1 h)^{n-1} Q + \frac{n(n-1)}{1.2}(u + u_1 h)^{n-2} Q^2 + \ldots . + Q^n$$

vanish, we arrange their terms according to ascending values of m and μ. Thus, making the coefficient of h^{n+1} vanish, we find

$$nu_1^{n-1}(2.1) + n(n-1)u_1^{n-2}u(3.1) + \frac{n(n-1)}{1.2}u_1^{n-2}(3.2) + \ldots . + (n+1.n) = 0,$$

and similarly, making the coefficient of h^{n+2} vanish,

$$nu_1^{n-1}(3.1) + n(n-1)u_1^{n-2}u(4.1) + \frac{n(n-1)}{1.2}u_1^{n-3}(4.2) + \ldots . + (n+2.n) = 0.$$

So in general the equation obtained by making the coefficient of $h^{n+\kappa}$ vanish consists of a series of numerical multiples (which are independent of the value

of x) of $u_1^{n-\theta}u^{\theta-\eta}(\theta+x,\eta)$ where η has all values from 1 to θ inclusive, and θ all values from 1 to n inclusive. Hence, by elimination, we find

$$\begin{vmatrix} (2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) & (5.1) & (5.2) & (5.3) & (5.4)\ldots \\ (3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) & (6.1) & (6.2) & (6.3) & (6.4)\ldots \\ (4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) & (7.1) & (7.2) & (7.3) & (7.4)\ldots \\ (5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) & (8.1) & (8.2) & (8.3) & (8.4)\ldots \\ (6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) & (9.1) & (9.2) & (9.3) & (9.4)\ldots \\ (7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3) & (10.1) & (10.2) & (10.3) & (10.4)\ldots \\ (8.1) & (9.1) & (9.2) & (10.1) & (10.2) & (10.3) & (11.1) & (11.2) & (11.3) & (11.4)\ldots \\ (9.1) & (10.1) & (10.2) & (11.1) & (11.2) & (11.3) & (12.1) & (12.2) & (12.3) & (12.4)\ldots \\ (10.1) & (11.1) & (11.2) & (12.1) & (12.2) & (12.3) & (13.1) & (13.2) & (13.3) & (13.4)\ldots \\ (11.1) & (12.1) & (12.2) & (13.1) & (13.2) & (13.3) & (14.1) & (14.2) & (14.3) & (14.4)\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

where the determinant on the left-hand side, consisting of $\frac{n^2+n}{2}$ rows and columns, is the Criterion of the curve of the n^{th} order.

Thus in the case of the Cubic Criterion, which we shall specially consider, we have $n=3$, and the elimination of $3u_1^2$, $6u_1u$, $3u_1$, $3u^2$, $3u$ and 1 between the six equations

$$\begin{aligned} 3u_1^2(2.1)+6u_1u(3.1)+3u_1(3.2)+3u^2(4.1)+3u(4.2)+(4.3)&=0,\\ 3u_1^2(3.1)+6u_1u(4.1)+3u_1(4.2)+3u^2(5.1)+3u(5.2)+(5.3)&=0,\\ 3u_1^2(4.1)+6u_1u(5.1)+3u_1(5.2)+3u^2(6.1)+3u(6.2)+(6.3)&=0,\\ 3u_1^2(5.1)+6u_1u(6.1)+3u_1(6.2)+3u^2(7.1)+3u(7.2)+(7.3)&=0,\\ 3u_1^2(6.1)+6u_1u(7.1)+3u_1(7.2)+3u^2(8.1)+3u(8.2)+(8.3)&=0,\\ 3u_1^2(7.1)+6u_1u(8.1)+3u_1(8.2)+3u^2(9.1)+3u(9.2)+(9.3)&=0, \end{aligned}$$

gives the Cubic Criterion in the form of the determinant

$$\begin{vmatrix} (2.1) & (3.1) & (3.2) & (4.1) & (4.2) & (4.3) \\ (3.1) & (4.1) & (4.2) & (5.1) & (5.2) & (5.3) \\ (4.1) & (5.1) & (5.2) & (6.1) & (6.2) & (6.3) \\ (5.1) & (6.1) & (6.2) & (7.1) & (7.2) & (7.3) \\ (6.1) & (7.1) & (7.2) & (8.1) & (8.2) & (8.3) \\ (7.1) & (8.1) & (8.2) & (9.1) & (9.2) & (9.3) \end{vmatrix}.$$

Remembering that

$$(m.\mu) = \text{co. } h^m \text{ in } (ah^2+bh^3+ch^4+\ldots.)^\mu,$$

it is easy to express the Criterion explicitly in terms of $a, b, c, \ldots.$

Thus, since

$$(ah^2+bh^3+ch^4+\ldots.)^2 = a^2h^4+2abh^5+(2ac+b^2)h^6+(2ad+2bc)h^7 \\ +(2ae+2bd+c^2)h^8+(2af+2be+2cd)h^9+\ldots.$$

and

$$(ah^2 + bh^3 + ch^4 + \ldots.)^3 = a^3h^6 + 3a^2bh^7 + (3a^2c + 3ab^2)h^8 + (3a^2d + 6abc + b^3)h^9 + \ldots.,$$

the Cubic Criterion may be written in the form

$$\begin{vmatrix} a & b & 0 & c & a^2 & 0 \\ b & c & a^2 & d & 2ab & 0 \\ c & d & 2ab & e & 2ac + b^2 & a^3 \\ d & e & 2ac + b^2 & f & 2ad + 2bc & 3a^2b \\ e & f & 2ad + 2bc & g & 2ae + 2bd + c^2 & 3a^2c + 3ab^2 \\ f & g & 2ae + 2bd + c^2 & h & 2af + 2be + 2cd & 3a^2d + 6abc + b^3 \end{vmatrix}$$

in which it was originally obtained by Mr. Roberts.

M. Halphen has remarked that the minor of h in the Cubic Criterion is the Principiant which he calls Δ (our $AC - B^2$) multiplied by a (see p. 50 of his Thèse).

We proceed to determine the degree and weight of the Criterion of the curve of the n^{th} order. These are the same as the degree and weight of its diagonal

$$(2.1)(4.1)(5.2)(7.1)(8.2)(9.3)(11.1)(12.2)(13.3)(14.4)\ldots.,$$

which consists of $\frac{n^2+n}{2}$ factors, separable into n groups,

$$(2.1),\ (4.1)(5.2),\ (7.1)(8.2)(9.3),\ (11.1)(12.2)(13.3)(14.4),\ \ldots.$$

containing $1, 2, 3, 4, \ldots. n$ factors respectively. Now,

$$(m.\mu) = \text{co. } h^m \text{ in } (ah^2 + bh^3 + ch^4 + \ldots.)^\mu$$
$$= \text{co. } h^{m-2\mu} \text{ in } (a + bh + ch^2 + \ldots.)^\mu,$$

and consequently $(m.\mu)$ is of degree μ and weight $m - 2\mu$. Hence the degree of the Criterion (found by adding together the second numbers of the duads which occur in the diagonal) is

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \ldots. + (1+2+3+\ldots.+n)$$
$$= 1 + 3 + 6 + 10 + \ldots. + \frac{n^2+n}{2}$$
$$= \frac{n(n+1)(n+2)}{6}.$$

To find the weight of the Criterion, we begin by arranging the factors of its diagonal according to their weight. This is done by writing each group of factors in reverse order, so that the diagonal is written thus:

$$(2.1)(5.2)(4.1)(9.3)(8.2)(7.1)(14.4)(13.3)(12.2)(11.1)\ldots.$$

The weights of the factors are now seen to be $0, 1, 2, 3, \ldots \frac{n^2+n}{2} - 1$; there being $\frac{n^2+n}{2}$ factors in the diagonal, one of them of zero weight. Hence the weight of the Criterion is

$$1 + 2 + 3 + \ldots + \left(\frac{n^2+n}{2} - 1\right)$$
$$= \frac{\left(\frac{n^2+n}{2} - 1\right)\frac{n^2+n}{2}}{2} = \frac{(n-1)\,n\,(n+1)(n+2)}{8}.$$

If, in the above formulae, we make $n = 2$, we shall find that the degree is 4 and the weight 3, whereas the Mongian $a^2d - 3abc + 2b^3$ (which is the Criterion of the second order) is of degree 3 and weight 3.

To account for this discrepancy, observe that in this case

$$\begin{vmatrix} (2.1) & (3.1) & (3.2) \\ (3.1) & (4.1) & (4.2) \\ (4.1) & (5.1) & (5.2) \end{vmatrix} = \begin{vmatrix} a & b & 0 \\ b & c & a^2 \\ c & d & 2ab \end{vmatrix},$$

which is divisible by a, the other factor being the Mongian, as may easily be verified. This is the only case in which the determinant expression for the Criterion contains an irrelevant factor.

To express the Cubic Criterion in terms of a, A, B, C, D, E, we first remark that its degree is $\frac{3.4.5}{6} = 10$, and its weight $\frac{2\ 3.4.5}{8} = 15$. Thus the Cubic Criterion is expressible as the product of $a^{-5}(\overset{\text{deg.}}{10} - \overset{\text{wt.}}{15} = -5)$ into a function of the capital letters, which we determine by the usual method of substituting for

$$\begin{array}{ccccccccc} a, & b, & c, & d, & e, & f, & g & , & h \quad , \\ 1, & 0, & 0, & A, & B, & C, & D + \frac{25}{8}A^2, & & E + \frac{15}{2}AB. \end{array}$$

When these substitutions are made, the Cubic Criterion becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & A & 0 & 0 \\ 0 & A & 0 & B & 0 & 1 \\ A & B & 0 & C & 2A & 0 \\ B & C & 2A & D + \frac{25}{8}A^2 & 2B & 0 \\ C & D + \frac{25}{8}A^2 & 2B & E + \frac{15}{2}AB & 2C & 3A \end{vmatrix}.$$

Subtracting the first column of this determinant from the fifth and reducing, we obtain

$$\begin{vmatrix} 0 & 1 & A & 0 & 0 \\ A & 0 & B & 0 & 1 \\ B & 0 & C & A & 0 \\ C & 2A & D+\frac{25}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & 2B & E+\frac{15}{2}AB & C & 3A \end{vmatrix}.$$

Again, subtracting the second column multiplied by A from the third and reducing, there results

$$-\begin{vmatrix} A & B & 0 & 1 \\ B & C & A & 0 \\ C & D+\frac{9}{8}A^2 & B & 0 \\ D+\frac{25}{8}A^2 & E+\frac{11}{2}AB & C & 3A \end{vmatrix},$$

which, after subtracting the first row multiplied by $3A$ from the last and reducing, becomes

$$\begin{vmatrix} B & C & A \\ C & D+\frac{9}{8}A^2 & B \\ D+\frac{1}{8}A^2 & E+\frac{5}{2}AB & C \end{vmatrix}$$

$$= B\left(CD+\frac{9}{8}A^2C - BE - \frac{5}{2}AB^2\right) + C\left(BD+\frac{1}{8}A^2B - C^2\right)$$
$$+ A\left(CE+\frac{5}{2}ABC - D^2 - \frac{5}{4}A^2D - \frac{9}{64}A^4\right)$$
$$= \left(ACE - B^2E - AD^2 + 2BCD - C^3\right) - \frac{5}{4}A\left(A^2D - 3ABC + 2B^3\right) - \frac{9}{64}A^5.$$

This expression, which is of degree-weight 15.15, instead of 10.15, must be divided by a^5 to give the correct value of the Cubic Criterion.

(*To be concluded in a subsequent number.*)